

Quantum energy inequalities and local covariance II: Categorical formulation

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Abstract

We formulate Quantum Energy Inequalities (QEIs) in the framework of locally covariant quantum field theory developed by Brunetti, Fredenhagen and Verch, which is based on notions taken from category theory. This leads to a new viewpoint on the QEIs, and also to the identification of a new structural property of locally covariant quantum field theory, which we call Local Physical Equivalence. Covariant formulations of the numerical range and spectrum of locally covariant fields are given and investigated, and a new algebra of fields is identified, in which fields are treated independently of their realisation on particular spacetimes and manifestly covariant versions of the functional calculus may be formulated.

1 Introduction

A very elegant formulation of local covariance for quantum field theory in curved spacetimes has been proposed recently by Brunetti, Fredenhagen and Verch [4] [hereafter abbreviated as BFV], utilising techniques from category theory. Ideas of this type have already played an important role in the proof of a rigorous spin–statistics connection in curved spacetimes [46], the perturbative renormalisation of interacting scalar field theories in curved spacetime [32, 33], and the theory of superselection sectors [6] and it seems that the complex of ideas set out by BFV will have many further applications in this area. (See also [5] for a review.) Indeed, one should say that *all* structural features of interest in QFT in CST should be formulated in this framework; any which are not capable of such reformulation must be either discarded as noncovariant, or (less likely, perhaps) prompt a review of the status of covariance itself.

This paper continues a discussion of the locally covariant aspects of quantum energy inequalities (QEIs) that was initiated in [24]. QEIs are the remnants in QFT of the classical energy conditions of general relativity (see [16, 44, 17] for recent reviews) and usually take the form of state-independent lower bounds on suitable averages of the stress-energy tensor. In [24], it was shown that known examples of QEIs can be formulated in a covariant fashion, and that this could be used to obtain a priori bounds on ground state energy densities in the Casimir effect and similar situations (see also [40]). The presentation in [24], while influenced by BFV, did not make use of the categorical formulation of local covariance, and it is the initial task of this paper to show how the gap may be bridged. The aim is to isolate the structures which might be characteristic of QEIs in general quantum field theories in curved spacetimes, with two ends in mind: first, as a preparatory step before attempting to derive QEIs in general covariant quantum field theories; second, so as to provide a framework for studying quantum field theories which are assumed to obey such bounds.

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We begin with a review of the BGV framework in Sect. 2, and then reexamine the definitions of locally covariant quantum energy inequalities given in [24] in this light. Guided by the mathematical framework, we are led to a more general viewpoint on such bounds, and our revised definitions encompass quantum fields other than the stress energy tensor, and permit state-dependent lower bounds. In fact, it turns out that state-dependent lower bounds are necessary to obtain QEIs on the non-minimally coupled scalar field [22] (see also Appendix B), and so the latter generalisation is not merely a mathematical extravagance. However, the new freedom would also permit rather trivial bounds, and we therefore give a first attempt at a definition of what a nontrivial quantum inequality should be. Inequalities on fields other than the stress-energy tensor are also of interest; see [39] for an application to squeezed states in quantum optics. We call these bounds quantum inequalities (QIs).

Investigation of our definitions quickly reveals the desirability of a new property of locally covariant quantum field theories, which we call *local physical equivalence*. This property, described in Sect. 4, ensures that no observer can tell, by finitely many local measurements made to finite tolerance, that the spacetime (s)he believes (s)he inhabits is not, in fact, part of a larger spacetime. This brings a more definite form to ideas expressed some time ago in Kay’s work on the Casimir effect [35] and a remark of Fulling in the appendix to [28], which have also played a role in the development of locally covariant QFT. In the situation at hand, local physical equivalence is needed to establish the covariance of various constructions relating to the QIs. We also show how information about a class of spacetimes with toroidal spatial sections can be used to make deductions about QIs in Minkowski space.

The discussion of quantum inequalities leads naturally to a broader consideration of the numerical range and spectrum of a locally covariant field, in Sect. 5. We study these objects partly for their own sake, but also because they suggest the utility of a new algebra of fields, abstracted from particular spacetimes or smearings. Algebras of this type offer manifestly covariant versions of constructions such as functional calculus and perhaps should be considered as the natural arena for the structural analysis of quantum fields in curved spacetime.

Although the present contribution is largely conceptual in scope, the ideas which led up to it have found concrete applications in providing the a priori bounds already mentioned [24, 40], and in proving the averaged null energy condition for the free scalar field along null geodesics with suitable Minkowskian neighbourhoods [21]. Moreover, the local physical equivalence property and the new abstract field algebras are of independent interest. While we do *not* seek to prove new QEIs in this paper, we show in Appendix A that the Wick square of the free scalar field obeys a locally covariant difference quantum inequality (by similar arguments to those expressed in [24]) and also establish a new result: namely that this difference quantum inequality is closely associated with a covariant absolute quantum inequality. A number of ideas for further study are summarised in the conclusion. Finally, Appendix B indicates the type of state-dependence that occurs in a QEI for the non-minimally coupled scalar field [22].

2 Categorical framework of locally covariant QFT

2.1 Categories, functors and natural transformations

To start, we briefly recall the definitions of some fundamental concepts from category theory, using, for the most part, the notation and terminology of [38]. First, a *category* \mathcal{C} consists of a set of objects $\text{obj } \mathcal{C}$, and, for every pair of objects A, B in \mathcal{C} , a set $\text{hom}_{\mathcal{C}}(A, B)$ of morphisms between A and B . A morphism in $\text{hom}_{\mathcal{C}}(A, B)$ is represented diagrammatically by $A \xrightarrow{f} B$ or $f : A \rightarrow B$. Every object $A \in \text{obj } \mathcal{C}$ has a unique identity morphism $\text{id}_A \in \text{hom}_{\mathcal{C}}(A, A)$; moreover, if $f : A \rightarrow B$ and $g : B \rightarrow C$ there is a composite morphism $g \circ f : A \rightarrow C$ obeying the unit law

$$\text{id}_B \circ f = f \quad g \circ \text{id}_B = g$$

and associativity, $(f \circ g) \circ h = f \circ (g \circ h)$. The category **Set** of small sets, with functions as morphisms, provides a standard example. Mention of ‘small sets’ is necessary here because the

collection of all sets is not a set, and therefore is too large to form a category according to our definition. Following Mac Lane [38, 37] we address foundational issues by assuming the existence of a single universe in addition to the ZFC axioms of set theory. The elements of the universe are called *small sets* and serve as the objects of ordinary mathematics, while subsets of the universe which are not also elements of it are referred to as *large sets*. The advantage is that even the large sets, and the universe itself, are sets within a model of ZFC set theory and so one is able to manage to a large extent without ever invoking proper classes or larger structures. Typically, the object sets of categories we study will be large sets [for example, obj Set is the universe] while the sets of homomorphisms between objects will be small.

Most of the categories we study will be *concrete*; that is, the objects are small sets (possibly with additional structure) and the morphisms are functions between them.

Turning to the second key concept, a *functor* \mathcal{F} between categories \mathbf{C} and \mathbf{C}' , written $\mathcal{F} : \mathbf{C} \rightarrow \mathbf{C}'$, is a map assigning to each object $A \in \text{obj } \mathbf{C}$ an object $\mathcal{F}(A) \in \mathbf{C}'$ and to each morphism $f \in \text{hom}_{\mathbf{C}}(A, B)$ a morphism $\mathcal{F}(f) \in \text{hom}_{\mathbf{C}'}(\mathcal{F}(A), \mathcal{F}(B))$ such that

$$\mathcal{F}(\text{id}_A) = \text{id}_{\mathcal{F}(A)}, \quad \mathcal{F}(f' \circ f) = \mathcal{F}(f') \circ \mathcal{F}(f)$$

for all $A \in \text{obj } \mathbf{C}$ and all composable morphisms f and f' . Functors are often described as covariant, in contradistinction to the notion of a *contravariant functor* $\mathcal{F} : \mathbf{C} \rightarrow \mathbf{C}'$, which assigns objects in \mathbf{C}' to objects in \mathbf{C} as before, but with the assignment of morphisms running in the opposite direction: to each $f \in \text{hom}_{\mathbf{C}}(A, B)$ the functor assigns a morphism $\mathcal{F}(f) \in \text{hom}_{\mathbf{C}'}(\mathcal{F}(B), \mathcal{F}(A))$, subject to the contravariance properties

$$\mathcal{F}(\text{id}_A) = \text{id}_{\mathcal{F}(A)}, \quad \mathcal{F}(f' \circ f) = \mathcal{F}(f) \circ \mathcal{F}(f').$$

Contravariant and covariant functors are related as follows. To each category \mathbf{C} there is an *opposite category*, \mathbf{C}^{op} , with the same objects as \mathbf{C} , but with all arrows reversed. That is, to each $f : A \rightarrow B$ in \mathbf{C} there is a unique $f^{\text{op}} : B \rightarrow A$ in \mathbf{C}^{op} , and every morphism in \mathbf{C}^{op} arises in this way.¹ Composition of morphisms in \mathbf{C}^{op} is given by the rule $f^{\text{op}} \circ g^{\text{op}} = (g \circ f)^{\text{op}}$ when the composition $g \circ f$ is defined in \mathbf{C} . Clearly $(\mathbf{C}^{\text{op}})^{\text{op}} = \mathbf{C}$. The assignments $A \mapsto A$, $f \mapsto f^{\text{op}}$ constitute a contravariant functor $\text{op}_{\mathbf{C}} : \mathbf{C} \rightarrow \mathbf{C}^{\text{op}}$; any covariant (resp., contravariant) functor $\mathcal{F} : \mathbf{C} \rightarrow \mathbf{C}'$ induces a contravariant (resp., covariant) functor $\mathcal{F}^{\text{op}} = \text{op}_{\mathbf{C}'} \circ \mathcal{F} : \mathbf{C} \rightarrow \mathbf{C}'^{\text{op}}$, and $(\mathcal{F}^{\text{op}})^{\text{op}} = \mathcal{F}$. We will make use of the freedom to replace contravariant functors by (covariant) functors later on. Note that, unless otherwise specified, functors are covariant.

We will also make use of the notion of a subfunctor. If \mathcal{F} and \mathcal{G} are covariant (resp., contravariant) functors from \mathbf{C} to \mathbf{C}' , where \mathbf{C}' is a concrete category, then \mathcal{F} is said to be a *subfunctor* of \mathcal{G} if (i) for each object A in \mathbf{C} , the object $\mathcal{F}(A)$ is a subset of $\mathcal{G}(A)$ and (ii) for each morphism $f : A \rightarrow B$ of \mathbf{C} , the morphism $\mathcal{F}(f)$ is the restriction of $\mathcal{G}(f)$ to $\mathcal{F}(A)$, denoted $\mathcal{F}(f) = \mathcal{G}(f)|_{\mathcal{F}(A)}$ (resp., the restriction to $\mathcal{F}(B)$, denoted $\mathcal{F}(f) = \mathcal{G}(f)|_{\mathcal{F}(B)}$).² In this case, we write $\mathcal{F} \subset \mathcal{G}$.³

The third concept is the idea of a natural transformation. Suppose \mathcal{F} and \mathcal{G} are functors between \mathbf{C} and \mathbf{C}' . A *natural transformation* between \mathcal{F} and \mathcal{G} , written $\tau : \mathcal{F} \rightarrow \mathcal{G}$, assigns to each object A of \mathbf{C} a morphism $\tau_A : \mathcal{F}(A) \rightarrow \mathcal{G}(A)$ in \mathbf{C}' , such that the rectangular part of the diagram

$$\begin{array}{ccccc} A & & \mathcal{F}(A) & \xrightarrow{\tau_A} & \mathcal{G}(A) \\ f \downarrow & & \mathcal{F}(f) \downarrow & & \downarrow \mathcal{G}(f) \\ B & & \mathcal{F}(B) & \xrightarrow{\tau_B} & \mathcal{G}(B) \end{array}$$

commutes whenever $f : A \rightarrow B$ in \mathbf{C} , i.e., $\tau_B \circ \mathcal{F}(f)$ and $\mathcal{G}(f) \circ \tau_A$ are identical morphisms in $\text{hom}_{\mathbf{C}'}(\mathcal{F}(A), \mathcal{G}(B))$.

¹Clearly $f^{\text{op}} : B \rightarrow A$ cannot necessarily be identified with a *function* from the underlying set of B to that of A , so \mathbf{C}^{op} need not be concrete, even if \mathbf{C} is.

²More precisely, \mathbf{C}' is concrete if there is a faithful functor (called the forgetful functor) from \mathbf{C}' to \mathbf{Set} , mapping any object of \mathbf{C}' to its underlying set, and any morphism to the underlying function. Here, a functor is faithful if its action on morphisms is injective. In defining subfunctors, one should strictly say that $U(\mathcal{F}(A)) \subset U(\mathcal{G}(A))$ and $U(\mathcal{F}(f))$ is the restriction of $U(\mathcal{G}(f))$, where U is the forgetful functor on \mathbf{C}' .

³As with set inclusions, the use of the symbol \subset is not intended to exclude equality.

2.2 Locally covariant quantum field theory

We may now describe the structure of locally covariant quantum field theory, largely following BFV but with some minor changes. This begins with a category \mathbf{Man} of spacetimes. More specifically, the objects of \mathbf{Man} are d -dimensional, oriented and time-oriented globally hyperbolic Lorentzian spacetimes, denoted \mathbf{M} , \mathbf{N} etc., where the notation denotes not just the underlying spacetime manifold, but also the specific choices of metric and (time-)orientation. Global hyperbolicity of \mathbf{M} requires strong causality and that $J_{\mathbf{M}}^+(p) \cap J_{\mathbf{M}}^-(q)$ is compact for all $p, q \in \mathbf{M}$ [30]. The morphisms of \mathbf{Man} are isometric embeddings $\psi : \mathbf{M} \rightarrow \mathbf{N}$ such that (i) $\psi(\mathbf{M})$ is an open globally hyperbolic subset⁴ of \mathbf{N} , and (ii) the (time-)orientation of \mathbf{M} coincides with that pulled back from \mathbf{N} via ψ .

A *locally covariant quantum field theory* is defined to be any functor $\mathcal{A} : \mathbf{Man} \rightarrow \mathbf{TAlg}$, the category of unital topological $*$ -algebras with continuous unit-preserving faithful $*$ -homomorphisms as morphisms. That is, \mathcal{A} assigns to each $\mathbf{M} \in \text{obj } \mathbf{Man}$ an algebra $\mathcal{A}(\mathbf{M})$, and to each morphism $\psi \in \text{hom}_{\mathbf{Man}}(\mathbf{M}, \mathbf{N})$ a faithful $*$ -homomorphism $\alpha_\psi = \mathcal{A}(\psi) \in \text{hom}_{\mathbf{TAlg}}(\mathcal{A}(\mathbf{M}), \mathcal{A}(\mathbf{N}))$, so that the covariance properties

$$\alpha_{\text{id}_{\mathbf{M}}} = \text{id}_{\mathcal{A}(\mathbf{M})} \quad \alpha_{\psi' \circ \psi} = \alpha_{\psi'} \circ \alpha_\psi$$

are obeyed, the latter holding whenever ψ' and ψ can be composed. We will follow BFV in writing α_ψ for $\mathcal{A}(\psi)$ to unburden the notation. Many variations are possible, for example, \mathbf{TAlg} could be replaced by the category of unital C^* -algebras, which is the main example in BFV.

The next step is the definition of a quantum field. We take a slightly more general viewpoint than BFV here, considering vector-valued fields which permit distributional smearings, instead of scalar fields smeared with smooth compactly supported test functions. This is necessary because QEIs are constraints on a rank-2 tensor field (the stress-energy tensor) in which distributional smearings are used to express averages along timelike submanifolds of the spacetime. Beginning with some notation, if $\mathbf{B} \xrightarrow{\pi} \mathbf{M}$ is a smooth vector bundle (with finite-dimensional fibre for simplicity) then the dual bundle (whose fibres are dual to those of \mathbf{B}) is denoted $\mathbf{B}^* \xrightarrow{\pi^*} \mathbf{M}$; moreover, $\mathcal{E}'(\mathbf{B})$ will denote the space of compactly supported ‘distributional sections’ of \mathbf{B} , i.e., the topological dual of $C^\infty(\mathbf{B}^*)$, the space of smooth sections of \mathbf{B}^* . The smooth vector bundles $\mathbf{B} \xrightarrow{\pi} \mathbf{M}$ over manifolds in \mathbf{Man} provide the objects of a category \mathbf{Bund} , in which the morphisms between $(\mathbf{B}_1, \pi_1, \mathbf{M}_1)$ and $(\mathbf{B}_2, \pi_2, \mathbf{M}_2)$ are pairs (ζ, ζ°) of smooth maps $\zeta : \mathbf{B}_1 \rightarrow \mathbf{B}_2$ and $\zeta^\circ \in \text{hom}_{\mathbf{Man}}(\mathbf{M}_1, \mathbf{M}_2)$ such that $\pi_2 \circ \zeta = \zeta^\circ \circ \pi_1$ and $\zeta|_{\pi_1^{-1}(x)} : \pi_1^{-1}(x) \rightarrow \pi_2^{-1}(\zeta^\circ(x))$ is a linear isomorphism for each $x \in \mathbf{M}_1$. We say that this bundle map *covers* the morphism ζ° . Note that our insistence that each $\zeta|_{\pi_1^{-1}(x)}$ is a linear isomorphism guarantees that \mathbf{B}_1 and \mathbf{B}_2 have a common fibre. A bundle morphism $(\zeta, \zeta^\circ) \in \text{hom}_{\mathbf{Bund}}(\mathbf{B}_1, \mathbf{B}_2)$ induces a push-forward ζ_* mapping $\mathcal{E}'(\mathbf{B}_1)$ to $\mathcal{E}'(\mathbf{B}_2)$ defined by

$$(\zeta_* f)(u) = f(\zeta^* u),$$

in terms of the pull-back of smooth sections $(\zeta^* u)(p) = \zeta|_{\pi_1^{-1}(p)}^*(u(\zeta^\circ(p)))$. The maps $\mathbf{B}_1 \rightarrow \mathcal{E}'(\mathbf{B}_1)$, $(\zeta, \zeta^\circ) \rightarrow \zeta_*$ constitute a functor $\mathcal{E}' : \mathbf{Bund} \rightarrow \mathbf{Set}$.⁵

To specify a quantum field, the first step is to give a functor $\mathcal{B} : \mathbf{Man} \rightarrow \mathbf{Bund}$ satisfying the requirement that, if $\psi : \mathbf{M} \rightarrow \mathbf{N}$, then $\mathcal{B}(\psi)$ should cover ψ . This functor determines the tensor or spinor type of the test fields. [These bundles might be associated bundles to a spin-bundle over \mathbf{M} , and, strictly speaking, one should include the choice of spin structure as part of the specification of \mathbf{M} and morphisms—see [46]; we have suppressed this here.] A *covariant set of smearing fields* is any subfunctor $\mathcal{D} \subset \mathcal{E}' \circ \mathcal{B}$: that is, each $\mathcal{D}(\mathbf{M})$ is a set of compactly supported distributional sections of the bundle $\mathcal{B}(\mathbf{M})$, and each morphism $\psi : \mathbf{M} \rightarrow \mathbf{N}$ in \mathbf{Man} has a push-forward action $\mathcal{D}(\psi) = \mathcal{B}(\psi)_*|_{\mathcal{D}(\mathbf{M})}$ injectively mapping $\mathcal{D}(\mathbf{M})$ into $\mathcal{D}(\mathbf{N})$. To unburden the notation, we write ψ_* for $\mathcal{D}(\psi)$. Each $\mathcal{D}(\mathbf{M})$ will be the class of test sections against which the quantum field (to be defined next) will be smeared on spacetime \mathbf{M} . A simple example is

⁴Since \mathbf{N} is globally hyperbolic, this amounts to the requirement that, for all $p, q \in \psi(\mathbf{M})$, each $J_{\mathbf{N}}^+(p) \cap J_{\mathbf{N}}^-(q)$ is contained in $\psi(\mathbf{M})$; an equivalent formulation (given in BFV) is that every causal curve in \mathbf{N} whose endpoints lie in $\psi(\mathbf{M})$ should be contained entirely in $\psi(\mathbf{M})$.

⁵In many circumstances it would be more natural to think of \mathcal{E}' as a functor to the category of topological vector spaces; however we wish to consider general subsets of testing functions in what follows, rather than subspaces.

provided by the scalar field, where we take $\mathcal{D}(\mathbf{M}) = C_0^\infty(\mathbf{M})$.⁶ On the other hand, Dimock's quantisation of the electromagnetic field [10] (see also [23]) provides an example where the set of smearing fields is restricted, in that case to the divergence-free one-forms on \mathbf{M} .

A *locally covariant quantum field* can now be described as a natural transformation $\Phi : \mathcal{D} \rightarrow \mathcal{A}$ between a locally covariant set of smearing fields and a locally covariant quantum field theory, represented by functors \mathcal{D} and \mathcal{A} as above, with \mathbf{TAlg} regarded as a subcategory of \mathbf{Set} .⁷ Namely, to each \mathbf{M} we associate a (not necessarily linear or continuous) map $\Phi_{\mathbf{M}} : \mathcal{D}(\mathbf{M}) \rightarrow \mathcal{A}(\mathbf{M})$ so that the rectangle in

$$\begin{array}{ccccc} \mathbf{M} & & \mathcal{D}(\mathbf{M}) & \xrightarrow{\Phi_{\mathbf{M}}} & \mathcal{A}(\mathbf{M}) \\ \psi \downarrow & & \psi_* \downarrow & & \downarrow \alpha_\psi \\ \mathbf{N} & & \mathcal{D}(\mathbf{N}) & \xrightarrow{\Phi_{\mathbf{N}}} & \mathcal{A}(\mathbf{N}) \end{array}$$

commutes, i.e., $\alpha_\psi(\Phi_{\mathbf{M}}(f)) = \Phi_{\mathbf{N}}(\psi_* f)$, where (as above) we have written ψ_* for the bundle morphism $\mathcal{D}(\psi)$.

This definition is a slight generalisation of that proposed by BFV, in that BFV only considered scalar fields (but see [46]) in the case where $\mathcal{D}(\mathbf{M})$ is the space of smooth compactly supported functions on \mathbf{M} , rather than a subset of the space of compactly supported distributional sections of a bundle. We have also formulated the natural transformation within \mathbf{Set} , rather than the category \mathbf{Top} of topological spaces; this amounts to dropping the continuity condition on $\Phi_{\mathbf{M}}$. On the other hand, BFV explicitly envisaged the possibility that $\Phi_{\mathbf{M}}$ might not be linear, so as to accommodate objects such as local S -matrices.

The final piece of general structure we will need is the concept of a locally covariant state space. If \mathcal{A} is a unital topological $*$ -algebra, its state space, denoted $\mathcal{A}_{+,1}^*$ is the convex set of positive ($\omega(A^*A) \geq 0$), normalised ($\omega(\mathbf{1}) = 1$) continuous linear functionals $\omega : \mathcal{A} \rightarrow \mathbb{C}$. We endow $\mathcal{A}_{+,1}^*$ with the weak- $*$ topology. The set of all states is generally too large for physical purposes, so it is convenient to refer to any convex subset $S \subset \mathcal{A}_{+,1}^*$, with the subspace topology, as a state space for \mathcal{A} . Such subsets will form the objects of a category \mathbf{States} . The morphisms in \mathbf{States} will be all continuous affine maps, i.e., $\mathcal{L} \in \text{hom}_{\mathbf{States}}(S, S')$ if $\mathcal{L} : S \rightarrow S'$ is continuous and obeys $\mathcal{L}(\lambda\omega + (1-\lambda)\omega') = \lambda\mathcal{L}\omega + (1-\lambda)\mathcal{L}\omega'$ for all $\omega, \omega' \in S$, $\lambda \in [0, 1]$. Our definitions differ slightly from those in BFV, who required that S should be closed under operations induced by \mathcal{A} and only considered morphisms which arise as duals of $*$ -algebra monomorphisms. The latter requirement will enter in our definition of the state space functor, so there is no essential difference in the present discussion.

Naturally associated with any functor $\mathcal{A} : \mathbf{Man} \rightarrow \mathbf{TAlg}$, there is a contravariant functor $\mathcal{A}_{+,1}^* : \mathbf{Man} \rightarrow \mathbf{States}$ given by $\mathcal{A}_{+,1}^*(\mathbf{M}) = \mathcal{A}(\mathbf{M})_{+,1}^*$ and $\mathcal{A}_{+,1}^*(\psi) = \alpha_\psi^*|_{\mathcal{A}_{+,1}^*(\mathbf{M})}$. We define a locally covariant *state space* for the theory \mathcal{A} to be any (contravariant) subfunctor $\mathcal{S} \subset \mathcal{A}_{+,1}^*$. Thus each $\mathcal{S}(\mathbf{M})$ is a convex subset of states on the algebra $\mathcal{A}(\mathbf{M})$ assigned to \mathbf{M} , and, for any $\psi : \mathbf{M} \rightarrow \mathbf{N}$ we have $\mathcal{S}(\psi) = \alpha_\psi^*|_{\mathcal{S}(\mathbf{N})}$, where $\alpha_\psi = \mathcal{A}(\psi)$ is the faithful $*$ -homomorphism between $\mathcal{A}(\mathbf{M})$ and $\mathcal{A}(\mathbf{N})$ induced by ψ .

The various structures introduced so far interact in the following way. Let Φ be a locally covariant quantum field associated with the locally covariant QFT \mathcal{A} and smearing fields \mathcal{D} . Suppose $\psi : \mathbf{M} \rightarrow \mathbf{N}$ in \mathbf{Man} and that $\omega \in \mathcal{S}(\mathbf{N})$. Then there is a state $\alpha_\psi^* \omega \in \mathcal{S}(\mathbf{M})$ with n -point function⁸

$$\alpha_\psi^* \omega(\Phi_{\mathbf{M}}(f_1) \cdots \Phi_{\mathbf{M}}(f_n)) = \omega(\alpha_\psi(\Phi_{\mathbf{M}}(f_1) \cdots \Phi_{\mathbf{M}}(f_n))) = \omega(\Phi_{\mathbf{N}}(\psi_* f_1) \cdots \Phi_{\mathbf{N}}(\psi_* f_n));$$

that is, the n -point function of $\alpha_\psi^* \omega$ on \mathbf{M} is the pull-back of the n -point function of ω on \mathbf{N} .

⁶Here, the underlying bundle is $\mathcal{B}(\mathbf{M}) = \mathbf{M} \times \mathbb{C}$ and $\mathcal{E}'(\mathbf{M} \times \mathbb{C})$ is identified with the usual class of compactly supported distributions on \mathbf{M} , into which $C_0^\infty(\mathbf{M})$ may be embedded using the metric-induced volume form on \mathbf{M} .

⁷More precisely, Φ is a natural transformation between \mathcal{D} and $U\mathcal{A}$, where $U : \mathbf{TAlg} \rightarrow \mathbf{Set}$ is the forgetful functor.

⁸We use the term slightly loosely in the situation where $\Phi_{\mathbf{M}}$ is not linear.

A key example to bear in mind is that of the Hadamard states of the free scalar field, which are distinguished by the wave-front set of the two-point function. Since the wave-front set transforms in a natural fashion under the pull-back of distributions, we indeed have the embedding $\alpha_\psi^* \mathcal{S}(\mathbf{N}) \subset \mathcal{S}(\mathbf{M})$.

3 Locally covariant quantum inequalities

3.1 Absolute quantum inequalities

The stress-energy tensor of classical matter is usually taken to obey certain *energy conditions*. For example, T_{ab} obeys the *weak energy condition* if $T_{ab}u^a u^b \geq 0$ for all timelike u^a , which means that all observers detect nonnegative energy density. It is not possible for such conditions to hold in quantum field theory at individual points [11]. In some models, however, local averages of the expectation value of the stress-energy tensor can be bounded from below as functions of the state, and these bounds constitute Quantum Energy Inequalities (QEIs). Inequalities of this type have been obtained for free fields in flat and curved spacetimes (see, e.g., [16, 44, 17] for discussion and references, and [26, 45] for recent results) and for two-dimensional conformal field theories in Minkowski space [18]. (A second type of QEI, to be discussed in the next section, has been proved in many more cases.) In [24] a notion of a locally covariant QEI was formulated without fully locating it within the categorical structures introduced by BFV; our purpose in this section is to remedy this and to explore some generalisations suggested in the process.

Suppose, then, that a quantum field theory, represented by a functor $\mathcal{A} : \mathbf{Man} \rightarrow \mathbf{TAlg}$ has a stress-energy tensor T . That is, there should be a functor $\mathcal{D} : \mathbf{Man} \rightarrow \mathbf{Set}$, with $C_0^\infty(T_0^2 \mathbf{M}) \subset \mathcal{D}(\mathbf{M}) \subset \mathcal{E}'(T_0^2 \mathbf{M})$, i.e., compactly supported distributional contravariant tensor fields of rank two, so that $T : \mathcal{D} \rightarrow \mathcal{A}$. (The field T should also satisfy conditions which identify it as the stress-energy tensor of the theory — see, e.g., the discussion in BFV — but we will not need these conditions here.) In [24], a locally covariant QEI was defined to be an assignment, to each $\mathbf{M} \in \mathbf{Man}$, of a subset $\mathcal{F}(\mathbf{M}) \subset \mathcal{E}'(T_0^2 \mathbf{M})$ and a function $\tilde{\mathcal{Q}}_{\mathbf{M}} : \mathcal{F}(\mathbf{M}) \rightarrow \mathbb{R}$ such that

$$\omega(T_{\mathbf{M}}(f)) \geq -\tilde{\mathcal{Q}}_{\mathbf{M}}(f) \quad (1)$$

for all $\omega \in \mathcal{S}(\mathbf{M})$; moreover, if $\psi : \mathbf{M} \rightarrow \mathbf{N}$ in \mathbf{Man} then $\psi_* \mathcal{F}(\mathbf{M}) \subset \mathcal{F}(\mathbf{N})$ and

$$\tilde{\mathcal{Q}}_{\mathbf{N}}(\psi_* f) = \tilde{\mathcal{Q}}_{\mathbf{M}}(f) \quad (2)$$

for all $f \in \mathcal{F}(\mathbf{M})$. Clearly, one would also require $\mathcal{F}(\mathbf{M}) \subset \mathcal{D}(\mathbf{M})$ if the latter is a proper subset of $\mathcal{E}'(T_0^2 \mathbf{M})$. More precisely, this was the definition of an *absolute* QEI, by contrast with the *difference* QEIs to be discussed later. Note that one needs the freedom to restrict the class of smearings $\mathcal{F}(\mathbf{M})$ because, even classically, not all smearings of the stress-energy tensor are expected to be semi-bounded.

Comparing this definition with the general structures described above, it is clear that the assignment $\mathbf{M} \mapsto \mathcal{F}(\mathbf{M})$, $\psi \mapsto \mathcal{F}(\psi) = \psi_*$ defines a functor $\mathcal{F} : \mathbf{Man} \rightarrow \mathbf{Set}$ which is a subfunctor of \mathcal{D} . In addition, Eq. (2) strongly suggests that each $\tilde{\mathcal{Q}}_{\mathbf{M}}$ is a component of a natural transformation between \mathcal{F} and some other functor from \mathbf{Man} to (a subcategory of) \mathbf{Set} . One way of making this precise is to define a constant functor $\mathcal{R} : \mathbf{Man} \rightarrow \mathbf{Set}$ by $\mathcal{R}(\mathbf{M}) = \mathbb{R}$ for all objects $\mathbf{M} \in \mathbf{Man}$ and $\mathcal{R}(\psi) = \text{id}_{\mathbb{R}}$ for all morphisms ψ of \mathbf{M} . Then the rectangular portion of

$$\begin{array}{ccccc} \mathbf{M} & \mathcal{F}(\mathbf{M}) & \xrightarrow{\tilde{\mathcal{Q}}_{\mathbf{M}}} & \mathbb{R} & = \mathcal{R}(\mathbf{M}) \\ \psi \downarrow & \psi_* \downarrow & & \downarrow \text{id}_{\mathbb{R}} & \\ \mathbf{N} & \mathcal{F}(\mathbf{N}) & \xrightarrow{\tilde{\mathcal{Q}}_{\mathbf{N}}} & \mathbb{R} & = \mathcal{R}(\mathbf{N}) \end{array}$$

commutes, so the $\tilde{\mathcal{Q}}_{\mathbf{M}}$ are indeed the components of a natural transformation $\tilde{\mathcal{Q}} : \mathcal{F} \rightarrow \mathcal{R}$.

However, it is known that there are quantum field theory models that do not obey QEIs of the above type. An example is given by the non-minimally coupled scalar field, for which it may be proved that the renormalised energy density is unbounded from below even after averaging [22]. The same is also expected for general interacting theories for reasons described in [41]. For these models, therefore, no $\tilde{\mathcal{Q}}_{\mathbf{M}}$ of the required type exists. We therefore broaden the notion of an absolute QEI to permit state-dependent lower bounds of the form

$$\omega(T_{\mathbf{M}}(f)) \geq -\omega(\mathcal{Q}_{\mathbf{M}}(f)) \quad \text{for all } \omega \in \mathcal{S}(\mathbf{M}),$$

where $\mathcal{Q}_{\mathbf{M}}$ is now an element of the algebra $\mathcal{A}(\mathbf{M})$. This must be supplemented by conditions that exclude trivial bounds (such as that obtained by setting $\mathcal{Q}_{\mathbf{M}}(f) = -T_{\mathbf{M}}(f)$); a first attempt at this will be described later. As before, covariance of the QEI is represented by the requirement that the $\mathcal{Q}_{\mathbf{M}}$ be components of a natural transformation, but now the transformation is between \mathcal{F} and \mathcal{A} , and is represented by the diagram

$$\begin{array}{ccc} \mathbf{M} & \mathcal{F}(\mathbf{M}) & \xrightarrow{\mathcal{Q}_{\mathbf{M}}} \mathcal{A}(\mathbf{M}) \\ \psi \downarrow & \psi_* \downarrow & \downarrow \alpha_\psi \\ \mathbf{N} & \mathcal{F}(\mathbf{N}) & \xrightarrow{\mathcal{Q}_{\mathbf{N}}} \mathcal{A}(\mathbf{N}). \end{array} \quad (3)$$

Note that a state-independent QEI of the form (1) is accommodated within our new definition by setting

$$\mathcal{Q}_{\mathbf{M}}(f) = \tilde{\mathcal{Q}}_{\mathbf{M}}(f)\mathbf{1}_{\mathcal{A}(\mathbf{M})};$$

because $\omega(\mathcal{Q}_{\mathbf{M}}(f)) = \tilde{\mathcal{Q}}_{\mathbf{M}}(f)$ for any state ω and, recalling that any α_ψ is unit-preserving, we have

$$\alpha_\psi(\mathcal{Q}_{\mathbf{M}}(f)) = \tilde{\mathcal{Q}}_{\mathbf{M}}(f)\alpha_\psi(\mathbf{1}_{\mathcal{A}(\mathbf{M})}) = \tilde{\mathcal{Q}}_{\mathbf{N}}(\psi_*f)\mathbf{1}_{\mathcal{A}(\mathbf{N})} = \mathcal{Q}_{\mathbf{N}}(\psi_*f)$$

whenever $\psi : \mathbf{M} \rightarrow \mathbf{N}$ in \mathbf{Man} .

We may now give our definition of a locally covariant absolute quantum inequality, which incorporates the above discussion and also allows for the possibility that fields other than the stress-energy tensor may be subject to similar bounds.

Definition 3.1 *Let Φ be a locally covariant quantum field associated with a locally covariant QFT \mathcal{A} and covariant set of smearing fields \mathcal{D} . A locally covariant absolute quantum inequality on Φ relative to a state space \mathcal{S} consists of a subfunctor $\mathcal{F} \subset \mathcal{D}$ and a natural transformation $\mathcal{Q} : \mathcal{F} \rightarrow \mathcal{A}$ such that*

$$\omega(\Phi_{\mathbf{M}}(f) + \mathcal{Q}_{\mathbf{M}}(f)) \geq 0$$

for all $\omega \in \mathcal{S}(\mathbf{M})$ and $f \in \mathcal{F}(\mathbf{M})$.

The naturality condition requires, of course, that

$$\alpha_\psi(\mathcal{Q}_{\mathbf{M}}(f)) = \mathcal{Q}_{\mathbf{N}}(\psi_*f)$$

for all $f \in \mathcal{F}(\mathbf{M})$, whenever $\psi : \mathbf{M} \rightarrow \mathbf{N}$ in \mathbf{Man} . We will sometimes use AQI for ‘absolute quantum inequality’.

Two remarks are in order. First, the continuity properties of the map $f \mapsto \mathcal{Q}_{\mathbf{M}}(f)$ are not fully understood in known examples of QEIs; this is why we chose to formulate the notion of a locally covariant field [and hence locally covariant QIs] without continuity assumptions, that is, in **Set**, rather than **Top**. In due course one might hope for a more finely grained definition.

Second, as already mentioned, we do not assume that the elements $\mathcal{Q}_{\mathbf{M}}(f) \in \mathcal{A}(\mathbf{M})$ are scalar multiples of the identity, in contrast to the usual QEI literature. The situation where $\mathcal{Q}_{\mathbf{M}}(f)$ is proportional to the identity is evidently very attractive, because it produces state-independent lower bounds. However, our more flexible definition has the advantage of encompassing theories such as the non-minimally coupled scalar field, which do not obey state-independent QEIs, but do

obey more general bounds [22] as we will briefly discuss in Appendix B. In addition, our definition has a natural expression in terms of an order relation among the fields of the theory – see Sect. 5. Having said this, we should clearly place some restrictions on \mathcal{Q} : for example, taking $\mathcal{F} = \mathcal{D}$ and $\mathcal{Q}_{\mathbf{M}}(f) = -\Phi_{\mathbf{M}}(f)$ rather trivially satisfies the definition. More generally, a quantum inequality of the above type will be called trivial on \mathbf{M} if, for each $f \in \mathcal{F}(\mathbf{M})$, there are constants $C_{\mathbf{M},f}$ and $C'_{\mathbf{M},f}$ (with possibly different engineering dimensions) such that

$$|\omega(\Phi_{\mathbf{M}}(f))| \leq C_{\mathbf{M},f} |\omega(\mathcal{Q}_{\mathbf{M}}(f))| + C'_{\mathbf{M},f} \quad (4)$$

for all $\omega \in \mathcal{S}(\mathbf{M})$. In this case, of course, there is nothing special about \mathcal{Q} as a *lower* bound. In section 4 we will prove that our definition of triviality is respected by local covariance (subject to the condition of Local Physical Equivalence, introduced below).

To illustrate the content of our definition of (non)triviality, note first that state-independent QIs (for which the right-hand side of (4) is independent of ω) are always nontrivial unless $\Phi_{\mathbf{M}}(f)$ has bounded expectation values on $\mathcal{S}(\mathbf{M})$. More generally, one might attempt to introduce a notion of the ‘order’ of a field $\Phi_{\mathbf{M}}$ as the infimum of those $\alpha \in \mathbb{R}^+$ so that for each $f \in \mathcal{D}(\mathbf{M})$ there exist constants $C_{\mathbf{M},f}$, $C'_{\mathbf{M},f}$ such that

$$|\omega(\Phi_{\mathbf{M}}(f))| \leq C_{\mathbf{M},f} (\mathcal{W}_{\mathbf{M},f}(\omega))^\alpha + C'_{\mathbf{M},f}$$

for all ω in (some suitable subset of) $\mathcal{S}(\mathbf{M})$, where $\mathcal{W}_{\mathbf{M},f}(\omega)$ is a measure of the energy content⁹ of the support of f or its immediate surroundings in state ω ; we assume $\mathcal{W}_{\mathbf{M},f}(\omega)$ can become arbitrarily large as ω varies. Fields that are scalar multiples of the identity would have order zero according to this definition. If a field can be bounded from below by a field of lower order, this would constitute a nontrivial quantum inequality according to our previous definition. One could also sharpen the definition of nontriviality in order to require the converse, i.e., that the lower bound must be of lower order to count as nontrivial. However a full investigation of the concept of ‘order’ sketched above goes beyond the scope of this paper, and we will work with the original definition of nontriviality here. We will return briefly to these issues in the conclusion.

As a separate example, although not in the covariant framework, consider an operator of the form $T = \sum_{i=1}^{\infty} \lambda_i a_i^* a_i$ on the usual Fock space with annihilation and creation operators obeying $[a_i, a_j^*] = \delta_{ij} \mathbf{1}$. If the λ_i are bounded from below, say by λ_0 , then we have a QI

$$\langle \psi | T \psi \rangle \geq \lambda_0 \langle \psi | N \psi \rangle$$

for all states ψ that are finite linear combinations of states created from the Fock vacuum by finitely many creation operators. Here $N = \sum_{i=1}^{\infty} a_i^* a_i$ is the usual number operator. This bound is nontrivial in the above sense if and only if the λ_i are unbounded from above.

As a digression, we observe that quantum inequalities, as we have defined them here, are strongly reminiscent of the Gårding inequalities arising in the study of pseudodifferential operators, in which one may obtain lower bounds ‘with a gain of two derivatives’. For example, a (nonzero) pseudodifferential operator of order m with a nonnegative symbol can be bounded from below by an operator of order $m - 2$ but cannot be bounded from above in this way [13]. Moreover, Gårding inequalities are closely related to the uncertainty principle, and provide a class of quantum mechanical quantum inequalities [12]. It is tempting to speculate that this link might run more deeply, and might indeed suggest an approach to quantum inequalities via the phase space properties of quantum field theory, in which the ‘gain of derivatives’ can be explicitly linked to nontriviality of the QI. Further evidence and comments in this direction may be found in [17, 20]. It also supports the contention that our definition of quantum inequalities is natural from a mathematical viewpoint.

Finally, it is important to mention two examples of locally covariant absolute quantum energy inequalities. First, Flanagan’s bound [27] for massless scalar fields in two dimensions is covariant in this sense [24]; second, the scalar field in four dimensions admits locally covariant absolute QEIs [26] and the same is expected for other free field theories, e.g., by combining the absolute QEI for the Dirac field obtained in [45] with the discussion of covariance in [9, 26].

⁹This might be based on local thermodynamic observables, cf. [7, 8].

3.2 Difference quantum inequalities

Much of the literature on QEIs concentrates on so-called *difference* inequalities, rather than the absolute bounds just discussed. In the state-independent case, which has been the main focus of the literature, difference quantum inequalities are statements of the form

$$\omega(\Phi_{\mathbf{M}}(f)) - \omega_0(\Phi_{\mathbf{M}}(f)) \geq -\tilde{\mathcal{Q}}_{\mathbf{M}}(f, \omega_0), \quad (5)$$

required to hold for a class of sampling functions f and states ω and ω_0 . Here, ω_0 is known as the reference state and ω as the state of interest. Historically, these bounds proved the easiest to obtain in curved spacetimes. As we have already allowed absolute QIs to depend on the state of interest, we should extend the same freedom to difference QIs. Thus, for our purposes, a difference QI will be a bound of the form

$$\omega(\Phi_{\mathbf{M}}(f)) - \omega_0(\Phi_{\mathbf{M}}(f)) \geq -\omega(\mathcal{Q}_{\mathbf{M}}(f, \omega_0)),$$

holding for all f , ω and ω_0 in appropriate classes.

To formulate difference QIs in categorical terms, we introduce an additional concept. The product $\mathbf{C}_1 \times \mathbf{C}_2$ of categories \mathbf{C}_1 and \mathbf{C}_2 has objects which are pairs $\langle A_1, A_2 \rangle$ of objects $A_i \in \text{obj } \mathbf{C}_i$. Morphisms between $\langle A_1, A_2 \rangle$ and $\langle B_1, B_2 \rangle$ are pairs $\langle f_1, f_2 \rangle$ of morphisms $f_i \in \text{hom}_{\mathbf{C}_i}(A_i, B_i)$; composition of morphisms being defined by $\langle f_1, f_2 \rangle \circ \langle f'_1, f'_2 \rangle = \langle f_1 \circ f'_1, f_2 \circ f'_2 \rangle$. Moreover, given functors $\mathcal{F}_i : \mathbf{C} \rightarrow \mathbf{C}_i$, we obtain a functor $\langle \mathcal{F}_1, \mathcal{F}_2 \rangle : \mathbf{C} \rightarrow \mathbf{C}_1 \times \mathbf{C}_2$ by

$$\langle \mathcal{F}_1, \mathcal{F}_2 \rangle(A) = \langle \mathcal{F}_1(A), \mathcal{F}_2(A) \rangle \quad \langle \mathcal{F}_1, \mathcal{F}_2 \rangle(f) = \langle \mathcal{F}_1(f), \mathcal{F}_2(f) \rangle$$

on objects and morphisms respectively.

As a particular example, in a locally covariant quantum field theory $\Phi : \mathcal{D} \rightarrow \mathcal{A}$ with state space \mathcal{S} , the functor $\langle \mathcal{D}, \mathcal{S}^{\text{op}} \rangle : \mathbf{Man} \rightarrow \mathbf{Set} \times \mathbf{States}^{\text{op}}$ maps each manifold \mathbf{M} to the pair $\langle \mathcal{D}(\mathbf{M}), \mathcal{S}(\mathbf{M}) \rangle$ and any morphism $\psi : \mathbf{M} \rightarrow \mathbf{N}$ to the pair of morphisms $\langle \psi_*, \alpha_\psi^{*\text{op}} \rangle$. We may now give our formal definition.

Definition 3.2 *Let Φ be a locally covariant quantum field associated with a locally covariant QFT \mathcal{A} and covariant set of smearing fields \mathcal{D} . A locally covariant difference quantum inequality on Φ relative to a state space \mathcal{S} consists of a subfunctor $\mathcal{F} \subset \mathcal{D}$ and a natural transformation $\mathcal{Q} : \langle \mathcal{F}, \mathcal{S}^{\text{op}} \rangle \rightarrow \mathcal{A}$ such that*

$$\omega(\Phi_{\mathbf{M}}(f)) - \omega_0(\Phi_{\mathbf{M}}(f)) \geq -\omega(\mathcal{Q}_{\mathbf{M}}(f, \omega_0)), \quad (6)$$

holds for all $f \in \mathcal{F}(\mathbf{M})$, $\omega, \omega_0 \in \mathcal{S}(\mathbf{M})$.

Here, naturality requires that the rectangle in

$$\begin{array}{ccc} \mathbf{M} & \langle \mathcal{F}(\mathbf{M}), \mathcal{S}(\mathbf{M}) \rangle & \xrightarrow{\mathcal{Q}_{\mathbf{M}}} \mathcal{A}(\mathbf{M}) \\ \psi \downarrow & \langle \psi_*, \alpha_\psi^{*\text{op}} \rangle \downarrow & \downarrow \alpha_\psi \\ \mathbf{N} & \langle \mathcal{F}(\mathbf{N}), \mathcal{S}(\mathbf{N}) \rangle & \xrightarrow{\mathcal{Q}_{\mathbf{N}}} \mathcal{A}(\mathbf{N}) \end{array}$$

commutes, that is,

$$\alpha_\psi(\mathcal{Q}_{\mathbf{M}}(f, \alpha_\psi^* \omega_0)) = \mathcal{Q}_{\mathbf{N}}(\psi_* f, \omega_0).$$

We will sometimes use superscripts d and a to distinguish difference and absolute QIs, and abbreviate ‘difference quantum inequality’ as DQI.

Many of the comments made regarding absolute QIs apply here also. In particular, as mentioned above, one is often interested in the ‘state independent’ situation where $\mathcal{Q}_{\mathbf{M}}(f, \omega_0) = \tilde{\mathcal{Q}}_{\mathbf{M}}(f, \omega_0) \mathbf{1}_{\mathcal{A}(\mathbf{M})}$, and the naturality requirement becomes

$$\tilde{\mathcal{Q}}_{\mathbf{M}}(f, \alpha_\psi^* \omega_0) = \tilde{\mathcal{Q}}_{\mathbf{N}}(\psi_* f, \omega_0),$$

which was the definition adopted in [24] for a locally covariant difference QI. Setting $\omega_0 = \omega$ in Eq. (6), it is clear that $\tilde{\mathcal{Q}}_M(f, \omega) \geq 0$ for all $f \in \mathcal{F}(M)$, $\omega \in \mathcal{S}(M)$.

It is also clear that one may have rather trivial difference QIs, such as that obtained by setting $\mathcal{Q}_M(f, \omega_0) = -\Phi_M(f) + \omega_0(\Phi_M(f))\mathbf{1}$. More generally, we define a difference QI to be trivial on M if, for all $f \in \mathcal{F}(M)$ and $\omega_0 \in \mathcal{S}(M)$, there exist constants C_{M,f,ω_0} and C'_{M,f,ω_0} such that

$$|\omega(\Phi_M(f))| \leq C_{M,f,\omega_0} |\omega(\mathcal{Q}_M(f, \omega_0))| + C'_{M,f,\omega_0}. \quad (7)$$

Again, all state-independent QIs are nontrivial. We also emphasise that difference QEIs conforming to our definition of local covariance are known [24].

There is a close relationship between difference and absolute QIs. In particular, given any AQI $\mathcal{Q}^a : \mathcal{F} \rightarrow \mathcal{A}$, define

$$\mathcal{Q}_M^d(f, \omega_0) = \mathcal{Q}_M^a(f) + \omega_0(\Phi_M(f))\mathbf{1}_{\mathcal{A}(M)} \quad (8)$$

for all $f \in \mathcal{F}(M)$, $\omega_0 \in \mathcal{S}(M)$ and $M \in \text{Man}$. Local covariance of \mathcal{Q}^d is easily checked: if $\psi : M \rightarrow N$ in Man then

$$\begin{aligned} \mathcal{Q}_N^d(\psi_* f, \omega_0) &= \mathcal{Q}_N^a(\psi_* f) + \omega_0(\Phi_N(\psi_* f))\mathbf{1}_{\mathcal{A}(N)} = \alpha_\psi(\mathcal{Q}_M^a(f) + \alpha_\psi^* \omega_0(\Phi_M(f))\mathbf{1}_{\mathcal{A}(M)}) \\ &= \alpha_\psi(\mathcal{Q}_M^d(f, \alpha_\psi^* \omega_0)), \end{aligned}$$

so $\mathcal{Q}^d : \langle \mathcal{F}, \mathcal{S}^{\text{op}} \rangle \rightarrow \mathcal{A}$. Moreover, \mathcal{Q}^d is a DQI as the elementary calculation

$$\omega(\Phi_M(f)) - \omega_0(\Phi_M(f)) \geq -\omega(\mathcal{Q}_M^a(f)) - \omega_0(\Phi_M(f)) = -\omega(\mathcal{Q}_M^d(f, \omega_0))$$

shows, using the fact that \mathcal{Q}^a is an AQI. One may also check that \mathcal{Q}^d , so defined, is a nontrivial DQI on spacetime M if and only if \mathcal{Q}^a is a nontrivial AQI on M .

Conversely, we may start with a locally covariant DQI \mathcal{Q}^d , and attempt to construct an AQI. At first sight this appears to be a simple matter of algebra: in any given spacetime M , and for any state $\omega_0 \in \mathcal{S}(M)$, we have

$$\omega(\Phi_M(f)) \geq -\omega(\mathcal{Q}_M^d(f, \omega_0) - \omega_0(\Phi_M(f))\mathbf{1}_{\mathcal{A}(M)})$$

holding for all $\omega \in \mathcal{S}(M)$, which suggests the simple rearrangement of (8)

$$\mathcal{Q}_M^a(f) = \mathcal{Q}_M^d(f, \omega_0) - \omega_0(\Phi_M(f))\mathbf{1}_{\mathcal{A}(M)}. \quad (9)$$

However, as noted by BFV, there is no way of covariantly specifying a single preferred state in every spacetime, so nontrivial dependence of the right-hand side on $\omega_0 \in \mathcal{S}(M)$ would present an obstruction to local covariance of the bound. Thus simple rearrangement allows one to pass from DQI to an AQI if and only if

$$\mathcal{Q}_M^d(f, \omega_0) - \mathcal{Q}_M^d(f, \omega_1) = (\omega_0(\Phi_M(f)) - \omega_1(\Phi_M(f)))\mathbf{1}_{\mathcal{A}(M)} \quad (10)$$

for all $f \in \mathcal{F}(M)$ and $\omega_0, \omega_1 \in \mathcal{S}(M)$. This is quite a strong condition, and it is remarkable that it holds for one of the main DQIs available in curved spacetimes, as we show in Appendix A. In general one would not have this independence, in which case the obvious approach is to take an infimum over $\omega_0 \in \mathcal{S}(M)$ in (9). Again covariance must be checked, and this turns out to need the new ingredient of local physical equivalence, to which we now turn.

4 Local physical equivalence

Our ability to probe physical systems with experiments is necessarily limited to a finite number of measurements made to finite tolerance. There is therefore good reason to regard two states as physically equivalent if they cannot be distinguished by tests of this type (see, for example, the lucid discussion in [1]). Similarly, two state spaces may be regarded as physically equivalent if the expectation values (of any finite set of observables) in any state in one may be arbitrarily

well-approximated by those corresponding to states in the other, and vice versa. Technically, this is equivalent to the two state spaces having equal closure in the weak-* topology on the set of all states.

Now consider a locally covariant quantum field theory $\mathcal{A} : \text{Man} \rightarrow \text{TAlg}$. If $\psi : \mathbf{M} \rightarrow \mathbf{N}$, the map $\alpha_\psi^*|_{\mathcal{S}(\mathbf{N})}$ sends each state $\omega \in \mathcal{S}(\mathbf{N})$ of the theory on \mathbf{N} to a state $\alpha_\psi^*\omega \in \mathcal{S}(\mathbf{M})$ of the theory on \mathbf{M} . However, there is no reason to suppose that this map is invertible, and indeed examples are known where it is not (see, e.g., the end of section II.B in [24]). Thus there can be ‘more’ states available to us on the spacetime \mathbf{M} than on the spacetime \mathbf{N} into which it is embedded.

However, the principle of locality should surely prevent us from determining, by local experiments, whether we truly live in \mathbf{M} , or on its image embedded in \mathbf{N} . Thus we should not be able to detect the ‘extra’ states on \mathbf{M} , which suggests the following requirement on the state space.

Definition 4.1 *\mathcal{S} respects local physical equivalence if, whenever $\psi : \mathbf{M} \rightarrow \mathbf{N}$, $\alpha_\psi^*\mathcal{S}(\mathbf{N})$ and $\mathcal{S}(\mathbf{M})$ have equal closures in the weak-* topology on $\mathcal{A}(\mathbf{M})^*$.*

This principle has not previously been identified in locally covariant quantum field theory and it is therefore necessary to check whether it holds in known models. To make a start, let us consider the situation in which each $\mathcal{A}(\mathbf{M})$ is a C^* -algebra, and each $\mathcal{S}(\mathbf{M})$ is closed under operations induced by $\mathcal{A}(\mathbf{M})$. That is, for any $\omega \in \mathcal{S}(\mathbf{M})$ and $A \in \mathcal{A}(\mathbf{M})$ for which $\omega(A^*A) > 0$, we have $\omega_A \in \mathcal{S}(\mathbf{M})$, where $\omega_A(B) = \omega(A^*BA)/\omega(A^*A)$. We will say that \mathcal{S} is closed under operations induced by \mathcal{A} in this case. This was the main focus in BFV. In this setting we have the following:

Lemma 4.2 *If \mathcal{S} is closed under operations induced by $\mathcal{A}(\mathbf{M})$, each $\mathcal{A}(\mathbf{M})$ is a C^* -algebra and each $\mathcal{S}(\mathbf{M})$ contains at least one state inducing a faithful GNS representation of $\mathcal{A}(\mathbf{M})$ then each $\mathcal{S}(\mathbf{M})$ is weak-* dense in the set of all states on $\mathcal{A}(\mathbf{M})$.*

Proof: Let $\varphi \in \mathcal{S}(\mathbf{M})$ induce a faithful representation of $\mathcal{A}(\mathbf{M})$, and suppose ω is an arbitrary state on $\mathcal{A}(\mathbf{M})$. By Fell’s theorem (Theorem 1.2 in [14]) states induced by finite rank density matrices in the GNS representation of φ are weak-* dense in $\mathcal{A}(\mathbf{M})_{+,1}^*$. Given any $\epsilon > 0$ then, we may find nonzero vectors $\psi_1, \dots, \psi_N \in \mathcal{H}_\varphi$ obeying $\sum_{r=1}^N \|\psi_r\|^2 = 1$ such that

$$\left| \omega(A) - \sum_{r=1}^N \langle \psi_r | \pi_\varphi(A) \psi_r \rangle \right| < \epsilon;$$

furthermore, because the GNS representation is cyclic, the ψ_r may be chosen, without loss of generality, to take the form $\psi_r = \pi_\varphi(A_r)\Omega_\varphi$ for $A_1, \dots, A_N \in \mathcal{A}(\mathbf{M})$. The sum in the last equation can be reexpressed as $\omega'(A)$, where $\omega' = \sum_{r=1}^N \varphi(A_r^*A_r)\varphi_{A_r}$. As $\sum_{r=1}^N \varphi(A_r^*A_r) = \sum_{r=1}^N \|\psi_r\|^2 = 1$, ω' is a finite convex combination of states obtained from $\varphi \in \mathcal{S}(\mathbf{M})$ by operations in $\mathcal{A}(\mathbf{M})$, and therefore belongs to $\mathcal{S}(\mathbf{M})$. Accordingly, ω is a weak-* limit of states in $\mathcal{S}(\mathbf{M})$. \square

Proposition 4.3 *Consider a locally covariant quantum field theory in which each $\mathcal{A}(\mathbf{M})$ is a C^* -algebra and \mathcal{S} is closed under operations induced by \mathcal{A} . If each $\mathcal{S}(\mathbf{M})$ contains at least one state inducing a faithful GNS representation of $\mathcal{A}(\mathbf{M})$ then \mathcal{S} respects local physical equivalence. In particular, if each $\mathcal{A}(\mathbf{M})$ is simple and each $\mathcal{S}(\mathbf{M})$ is nonempty then \mathcal{S} respects local physical equivalence.*

Proof: It is sufficient to show that, for every morphism $\psi : \mathbf{M} \rightarrow \mathbf{N}$ in Man , $\mathcal{S}(\mathbf{M})$ is contained in the weak-* closure of $\alpha_\psi^*\mathcal{S}(\mathbf{N})$ (the reverse inclusion is trivial). Accordingly, let ω be any state in $\mathcal{S}(\mathbf{M})$. As α_ψ is a *-morphism of C^* -algebras, $\alpha_\psi(\mathcal{A}(\mathbf{M}))$ is a C^* -subalgebra of $\mathcal{A}(\mathbf{N})$ (Prop. 2.3.1 in [3]) on which ω induces a state in the obvious way. This state can be extended using the Hahn–Banach theorem to a state $\hat{\omega}$ on $\mathcal{A}(\mathbf{N})$ (Prop. 2.3.24 of [3]), with the property $\hat{\omega}(\alpha_\psi A) = \omega(A)$ for all $A \in \mathcal{A}(\mathbf{M})$. Of course there is no reason to suppose that $\hat{\omega} \in \mathcal{S}(\mathbf{N})$, but,

$\widehat{\omega}$ must be the weak-* limit of a sequence of states ω_n in $\mathcal{S}(\mathbf{N})$ by Lemma 4.2. This induces a sequence $\alpha_\psi^* \omega_n \in \mathcal{S}(\mathbf{M})$ such that

$$\alpha_\psi^* \omega_n(A) = \omega_n(\alpha_\psi A) \longrightarrow \widehat{\omega}(\alpha_\psi A) = \omega(A)$$

for all $A \in \mathcal{A}(\mathbf{M})$. Thus ω belongs to the weak-* closure of $\alpha_\psi^* \mathcal{S}(\mathbf{N})$. As ω was arbitrary the first part of the result is proved. The second part follows because all nontrivial representations of simple algebras are faithful. \square

We remark that this establishes local physical equivalence for nontrivial locally covariant free field theories in which each $\mathcal{A}(\mathbf{M})$ is a Weyl or CAR algebra. More generally, however, the lack of an analogue to Fell's theorem for general *-algebras renders local physical equivalence a nontrivial addition to the structure of locally covariant quantum field theory. One needs to check that—as one would expect—it does in fact hold for known models: it is planned to address this elsewhere [19].

Our focus now reverts to quantum inequalities. For the rest of this section, we will assume that Φ is a locally covariant field associated with a locally covariant QFT \mathcal{A} and test space \mathcal{D} . We will then make a number of consistency checks on the definitions set out above, each of which will turn out to use local physical equivalence. In addition, we will show how this principle may be used to infer constraints on AQIs on Minkowski space from information about a family of spacetimes with toroidal spatial topology.

To begin, suppose one has a sharp quantum inequality on each spacetime, for the largest class of sampling functions possible. Is it necessarily locally covariant? It would seem strange if covariance did not favour the best possible bounds, and would suggest that our definitions were defective. We analyse this issue for state-independent QIs. Accordingly, suppose that \mathcal{S} is a locally covariant state space for the theory. Define

$$\tilde{\mathcal{Q}}_M^a(f) = - \inf_{\omega \in \mathcal{S}(M)} \omega(\Phi_M(f))$$

for each $f \in \mathcal{D}(M)$ and set $\mathcal{Q}_M^a(f) = \tilde{\mathcal{Q}}_M^a(f) \mathbf{1}_{\mathcal{A}(M)}$ and

$$\mathcal{F}(M) = \{f \in \mathcal{D}(M) : \tilde{\mathcal{Q}}_M^a(f) < \infty\},$$

which, by construction, provides a sharp state-independent absolute QI on the largest class of sampling functions possible on each spacetime. (Of course, for some fields $\mathcal{F}(M)$ might be empty, or the bound might be trivial.)

We now proceed to analyse the covariance properties of this quantum inequality, supposing that $\psi : M \rightarrow N$. Since each $\omega \in \mathcal{S}(N)$ induces a state $\alpha_\psi^* \omega \in \mathcal{S}(M)$, we have

$$\begin{aligned} -\tilde{\mathcal{Q}}_M^a(f) &\leq \inf_{\omega \in \mathcal{S}(N)} \alpha_\psi^* \omega(\Phi_M(f)) = \inf_{\omega \in \mathcal{S}(N)} \omega(\alpha_\psi \Phi_M(f)) \\ &= \inf_{\omega \in \mathcal{S}(N)} \omega(\Phi_N(\psi_* f)) = -\tilde{\mathcal{Q}}_N^a(\psi_* f), \end{aligned} \tag{11}$$

from which we may conclude that $\psi_* \mathcal{F}(M) \subset \mathcal{F}(N)$ so we have a morphism $\mathcal{F}(\psi) = \psi_*|_{\mathcal{F}(M)}$ from $\mathcal{F}(M)$ to $\mathcal{F}(N)$ in **Set**. It is obvious that composition and the identity property hold, so in fact \mathcal{F} is a functor as required.

However, \mathcal{Q}^a is not a natural transformation unless the inequality in (11) can be replaced by an equality. This hiatus may be resolved provided that \mathcal{S} respects local physical equivalence. By definition, given any $\epsilon > 0$ one may approximate $-\tilde{\mathcal{Q}}_M^a(f)$ to within $\epsilon/2$ by the expectation value of $\Phi_M(f)$ in some state $\omega \in \mathcal{S}(M)$. Using local physical equivalence, $\omega(\Phi_M(f))$ may itself be approximated to within $\epsilon/2$ by $\alpha_\psi^* \omega'(\Phi_M(f))$ for some state $\omega' \in \mathcal{S}(N)$. But

$$\alpha_\psi^* \omega'(\Phi_M(f)) = \omega'(\Phi_N(\psi_* f)) \geq -\tilde{\mathcal{Q}}_N^a(\psi_* f)$$

using covariance and the QI on N . Hence $-\tilde{\mathcal{Q}}_M^a(f) + \epsilon \geq -\tilde{\mathcal{Q}}_N^a(\psi_* f)$, and since ϵ was arbitrary we may conclude, putting this together with (11), that (2) now holds. Therefore \mathcal{Q}^a is natural and our absolute QI (though possibly trivial) is locally covariant. We may formulate this as follows.

Proposition 4.4 *Suppose Φ is any locally covariant quantum field, associated with the functors \mathcal{A} and \mathcal{D} and a state space \mathcal{S} which respects local physical equivalence. The sharp state-independent absolute QI on Φ , relative to \mathcal{S} , defined on the largest class of test functions possible on each globally hyperbolic spacetime, is automatically locally covariant.*

A sharp difference QI on Φ may be defined in a very similar way, by setting

$$\tilde{\mathcal{Q}}_M^d(f, \omega_0) = - \inf_{\omega \in \mathcal{S}(M)} (\omega(\Phi_M(f)) - \omega_0(\Phi_M(f))) .$$

It is obvious that this is defined on the same set $\mathcal{F}(M)$ as the absolute QI obtained above, and moreover that

$$\tilde{\mathcal{Q}}_M^d(f, \omega_0) = \tilde{\mathcal{Q}}_M^a(f) + \omega_0(\Phi_M(f)) .$$

The reverse construction is also of interest. Suppose one is given a locally covariant state-independent difference QI (not necessarily sharp). Does there exist a locally covariant absolute QI on Φ ? The answer to this is affirmative, subject to local physical equivalence: first rearrange the basic difference inequality as

$$\tilde{\mathcal{Q}}_M^d(f, \omega') - \omega'(\Phi_M(f)) \geq -\omega(\Phi_M(f)) , \quad (12)$$

where ω' is the reference state and $\omega \in \mathcal{S}(M)$ is arbitrary. But now fix ω and allow ω' to vary. Since the left-hand side is clearly bounded from below,

$$\tilde{\mathcal{Q}}_M^a(f) \stackrel{\text{def}}{=} \inf_{\omega' \in \mathcal{S}(M)} \left(\tilde{\mathcal{Q}}_M^d(f, \omega') - \omega'(\Phi_M(f)) \right) \quad (13)$$

is finite for all $f \in \mathcal{F}(M)$, and independent of ω . Moreover, from (12)

$$\omega(\Phi_M(f)) \geq -\tilde{\mathcal{Q}}_M^a(f) .$$

As ω was arbitrary, we obtain an absolute QI by setting $\mathcal{Q}_M^a(f) = \tilde{\mathcal{Q}}_M^a(f)\mathbf{1}$.

To establish local covariance, we assume in addition that $\omega \mapsto \tilde{\mathcal{Q}}_M^d(f, \omega)$ is weak-* continuous for each f , and then proceed along lines similar to those used before. Because the domain of sampling functions is unchanged, \mathcal{F} has the required properties; in addition we may calculate

$$\begin{aligned} \tilde{\mathcal{Q}}_M^a(f) &\leq \inf_{\omega' \in \mathcal{S}(N)} \left(\tilde{\mathcal{Q}}_M^d(f, \alpha_\psi^* \omega') - \alpha_\psi^* \omega'(\Phi_M(f)) \right) \\ &= \inf_{\omega' \in \mathcal{S}(N)} \left(\tilde{\mathcal{Q}}_N^d(\psi_* f, \omega') - \omega'(\Phi_N(\psi_* f)) \right) \\ &= \tilde{\mathcal{Q}}_N^a(\psi_* f) , \end{aligned}$$

so it is only necessary to show that the reverse inequality holds. This proceeds by first approximating $\tilde{\mathcal{Q}}_M^a(f)$ by some $\tilde{\mathcal{Q}}_M^d(f, \omega') - \omega'(\Phi_M(f))$ and then using local physical equivalence and weak-* continuity of $\omega' \mapsto \tilde{\mathcal{Q}}_M^d(f, \omega')$ to approximate this, in turn, by $\tilde{\mathcal{Q}}_M^d(f, \alpha_\psi^* \omega'') - \alpha_\psi^* \omega''(\Phi_M(f))$ for some $\omega'' \in \mathcal{S}(N)$. The remainder of the argument runs parallel to that given above, and need not be repeated. To summarise, we have established the following.

Proposition 4.5 *Suppose Φ is any locally covariant quantum field, associated with the functors \mathcal{A} and \mathcal{D} and a state space \mathcal{S} which respects local physical equivalence. Let \mathcal{Q}^d be any state-independent locally covariant difference QI on Φ relative to \mathcal{S} , with smearing fields \mathcal{F} such that $\mathcal{Q}_M^d(f, \omega) = \tilde{\mathcal{Q}}_M^d(f, \omega)\mathbf{1}_{\mathcal{A}(M)}$, with $\omega \mapsto \tilde{\mathcal{Q}}_M^d(f, \omega)$ weak-* continuous for each $f \in \mathcal{F}(M)$ and every M . Then equation (13) defines a state-independent, locally covariant, absolute QI on Φ , relative to \mathcal{S} , with the same smearing fields \mathcal{F} .*

In Appendix A, we will show that the hypothesis of weak-* continuity is satisfied for a DQI on the Wick square of the free scalar field, so Prop. 4.5 applies. In this case, however, the quantity inside the infimum in (13) is independent of ω' (as is also shown in Appendix A), so the bound obtained coincides with that obtained by rearrangement in Sect. 3.2. The Wick square DQI also obeys the hypotheses of part (b) of the following result, which demonstrates that our notion of a (non)trivial QI is compatible with local covariance.

Proposition 4.6 *Consider a locally covariant quantum field Φ associated with a theory respecting local physical equivalence. (a) Let $\mathcal{Q}^a : \mathcal{F} \rightarrow \mathcal{A}$ be a locally covariant absolute QI on Φ relative to \mathcal{S} . Suppose $\psi : M \rightarrow N$ in Man . If \mathcal{Q}^a is trivial on N then it is trivial on M . (Conversely, if \mathcal{Q}^a is nontrivial on M then it is nontrivial on N .) (b) Suppose $\mathcal{Q}^d : \langle \mathcal{F}, \mathcal{S}^{\text{op}} \rangle \rightarrow \mathcal{A}$ is a locally covariant DQI such that $\mathcal{S}(M) \times \mathcal{S}(M) \ni (\omega, \omega_0) \mapsto \omega(\mathcal{Q}_M(f, \omega_0))$ is weak-* continuous in ω_0 , uniformly in ω , for each $f \in \mathcal{F}(M)$ and every $M \in \text{Man}$. If $\psi : M \rightarrow N$ in Man and \mathcal{Q}^d is trivial on N , then \mathcal{Q}^d is trivial on M .*

Proof: (a) Choose any fixed positive constant c with the dimensions of $\omega(\mathcal{Q}_M^a(f))$. Then triviality on any spacetime M is equivalent to the statement that

$$\sup_{\omega \in \mathcal{S}(M)} \frac{|\omega(\Phi_M(f))|}{|\omega(\mathcal{Q}_M^a(f))| + c} < \infty \quad (14)$$

for all $f \in \mathcal{F}(M)$. Supposing that \mathcal{Q} is trivial on N , we may use covariance to observe that $\omega \mapsto |\omega(\Phi_M(f))|/(|\omega(\mathcal{Q}_M^a(f))| + c)$ is bounded on the set $\alpha_\psi^* \mathcal{S}(N)$. Now take an arbitrary $\omega \in \mathcal{S}(M)$; using local physical equivalence we may find a sequence $\omega_n \in \alpha_\psi^* \mathcal{S}(N)$ such that $\omega_n(\Phi_M(f)) \rightarrow \omega(\Phi_M(f))$ and $\omega_n(\mathcal{Q}_M(f)) \rightarrow \omega(\mathcal{Q}_M(f))$. Combining this with our first observation we see that (14) holds, so \mathcal{Q}^a is trivial on M .

The proof of part (b) is similar. Choosing any fixed constant c with the dimensions of $\omega(\mathcal{Q}_M(f, \omega_0))$, triviality on any spacetime M is equivalent to the statement that

$$\sup_{\omega \in \mathcal{S}(M)} \frac{|\omega(\Phi_M(f))|}{|\omega(\mathcal{Q}_M^d(f, \omega_0))| + c} < \infty$$

for all $\omega_0 \in \mathcal{S}(M)$ and $f \in \mathcal{F}(M)$. Using triviality on N , we may infer that the ratio $|\omega(\Phi_M(f))|/(|\omega(\mathcal{Q}_M^d(f, \omega_0))| + c)$ is bounded for all $\omega \in \alpha_\psi^* \mathcal{S}(N)$ for each fixed $\omega_0 \in \alpha_\psi^* \mathcal{S}(N)$. We use local physical equivalence to extend this to all $\omega \in \mathcal{S}(M)$, and then the uniform weak-* continuity hypothesis and local physical equivalence to extend again to all $\omega_0 \in \mathcal{S}(M)$. Hence \mathcal{Q}^d is trivial on M . \square

To conclude this section, we show how information on a class of spacetimes with toroidal spatial topology can be used to infer information about AQIs in Minkowski space. Consider a theory $\mathcal{A} : \text{Man} \rightarrow \text{TAlg}$ with a state space \mathcal{S} which respects local physical equivalence, and let $\Phi : \mathcal{D} \rightarrow \mathcal{A}$ be a field of the theory. To keep matters simple, we restrict to the situation where $\mathcal{D}(M) = C_0^\infty(M)$. We assume in addition that each Φ_M is a linear map, that $\Phi_M(f)^* = \Phi_M(\bar{f})$ for all $f \in \mathcal{D}(M)$ and that the one-point functions of Φ with respect to \mathcal{S} are smooth, i.e., for each $\omega \in \mathcal{S}(M)$ there is a smooth function $p \mapsto \omega(\Phi_M(p))$ on M such that

$$\omega(\Phi_M(f)) = \int_M \omega(\Phi_M(p)) f(p) d\text{vol}_M(p)$$

for all $f \in \mathcal{D}(M)$ and each $M \in \text{Man}$.

Let M_0 be n -dimensional Minkowski space, and fix a system of inertial coordinates (t, x_1, \dots, x_{n-1}) . Define N_L to be the quotient of M_0 by the group \mathbb{Z}^{n-1} generated by translations through proper distances L along the x_i -axes, so each N_L has the topology of $\mathbb{R} \times \mathbb{T}^{n-1}$. Suppose that each $\mathcal{S}(N_L)$ contains a set of translationally invariant states S_L and set

$$\kappa(L) = \inf_{\omega \in S_L} \omega(\Phi_{N_L}(p))$$

(which is independent of the particular $p \in N_L$, of course). In the typical situation of interest for QIs, the function κ is monotone increasing in L and $\kappa(L) \rightarrow -\infty$ as $L \rightarrow 0^+$. We will assume this in what follows. One final definition: given any subset S of M_0 , the *timelike diameter* of S is the infimum, over all points p, q for which $S \subset I^+(p) \cap I^-(q)$, of the interval between p and q .

Proposition 4.7 *For each nonnegative $f \in \mathcal{D}(M_0)$, we have*

$$\inf_{\omega \in \mathcal{S}(M_0)} \omega(\Phi_{M_0}(f)) \leq \kappa(\ell(f)) \int_{M_0} f d\text{vol}_{M_0},$$

where $\ell(f)$ is the timelike diameter of the support of f .

Proof: Let $p_{\pm} \in M_0$ be any points such that

$$\text{supp } f \subset D \stackrel{\text{def}}{=} I_{M_0}^+(p_-) \cap I_{M_0}^-(p_+)$$

and construct inertial coordinates $x' = (t', \mathbf{x}')$ in which p_{\pm} have coordinates $(\pm L/2, \mathbf{0})$, so that D is described by the inequality $|t'| + |\mathbf{x}'| < L/2$. Endowing D with the metric and (time)-orientation induced from M_0 , it becomes an object \mathbf{D} of Man in its own right, with a morphism $\iota : \mathbf{D} \rightarrow M_0$ given by the inclusion mapping. In addition there is a morphism $\psi : \mathbf{D} \rightarrow N_L$, which is the “smallest” of the N_L into which \mathbf{D} may be embedded (note that \mathbf{D} is open, so it is just too small to detect the topology of N_L).

Given any $\omega \in S_L$, the state $\alpha_{\psi}^* \omega \in \mathcal{S}(\mathbf{D})$ obeys

$$\begin{aligned} \alpha_{\psi}^* \omega(\Phi_{\mathbf{D}}(\iota^* f)) &= \omega(\Phi_{N_L}(\psi_* \iota^* f)) = \omega(\Phi_{N_L}(p)) \int_{N_L} \psi_* \iota^* f \, d\text{vol}_{N_L} \\ &= \omega(\Phi_{N_L}(p)) \int_{M_0} f \, d\text{vol}_{M_0}, \end{aligned}$$

where $p \in N_L$ is arbitrary. Given an arbitrary $\epsilon > 0$, therefore, there exists a state $\omega' \in \mathcal{S}(\mathbf{D})$ such that

$$\omega'(\Phi_{\mathbf{D}}(\iota^* f)) < (\kappa(L) + \epsilon) \int_{M_0} f \, d\text{vol}_{M_0}$$

and, by local physical equivalence, there exists $\omega'' \in \mathcal{S}(M_0)$ such that

$$|\omega''(\Phi_{M_0}(f)) - \omega'(\Phi_{\mathbf{D}}(\iota^* f))| < \epsilon \int_{M_0} f \, d\text{vol}_{M_0}.$$

Since all the expectation values are real and f is nonnegative, we have

$$\omega''(\Phi_{M_0}(\iota^* f)) < (\kappa(L) + 2\epsilon) \int_{M_0} f \, d\text{vol}_{M_0}$$

and so, taking $\epsilon \rightarrow 0^+$ and also optimising over all double cones containing $\text{supp } f$, we obtain the required result. \square

It follows immediately that any state independent QI on Φ , relative to the nonnegative functions in $\mathcal{D}(M)$, must obey

$$\tilde{\mathcal{Q}}_{M_0}^a(f) \geq -\kappa(\ell(f)) \int_{M_0} f \, d\text{vol}_{M_0},$$

so the scaling behaviour of κ bounds that of $\tilde{\mathcal{Q}}_{M_0}^a(f)$ if one shrinks the support of f while holding its integral constant. A similar situation occurs if the support of f converges to a null geodesic segment with endpoints q_{\pm} (with q_- to the past of q_+) because the double cones spanned by $p_{\pm} = \tau_{\pm\epsilon} q_{\pm}$ have timelike diameter of order ϵ as $\epsilon \rightarrow 0^+$, where τ_s is translation along a constant future-pointing timelike vector field. Hence $\tilde{\mathcal{Q}}_{M_0}^a(f_n) \rightarrow \infty$ for any sequence f_n with $\int_{M_0} f_n \, d\text{vol}_{M_0}$ fixed for all n , whose support shrinks onto a null geodesic segment. This is consistent with the facts that there are no QEIs for the free scalar field for averaging along finite portions of null curves [25] and (not unrelated) that there are not expected to be nontrivial observables localised on null geodesic segments in spacetimes of dimensions $d \geq 2$ (see [47] and footnotes 34 and 35 in [25]).

5 The covariant numerical range and spectrum

Let \mathcal{D} be any covariant set of smearing fields and \mathcal{A} a locally covariant quantum field theory. The set¹⁰ $\text{Nat}(\mathcal{D}, \mathcal{A})$ of natural transformations, i.e., those quantum fields of the model whose

¹⁰See below for a proof that this is in fact a small set.

smearings are drawn from \mathcal{D} , may be given a partial order as follows: we write $\Phi \geq \Psi$ if

$$\omega(\Phi_M(f)) \geq \omega(\Psi_M(f)) \quad \forall M \in \text{Man}, f \in \mathcal{D}(M), \omega \in \mathcal{S}(M). \quad (15)$$

Of course, if $\mathcal{D}(M)$ is a vector space, and Φ_M and Ψ_M are linear maps from $\mathcal{D}(M)$ to $\mathcal{A}(M)$, it cannot hold that $\Phi \geq \Psi$ unless Φ and Ψ are \mathcal{S} -indistinguishable, i.e., $\omega(\Phi_M(f)) = \omega(\Psi_M(f))$ for all $M \in \text{Man}$, $f \in \mathcal{D}(M)$ and $\omega \in \mathcal{S}(M)$, because the inequality in (15) must hold for both f and $-f$. Recall, however, that we do not require in general that fields are linear or that sets of smearing functions are vector spaces (this applies in particular to the restricted sets of smearing functions considered in QIs).

Quantum Inequalities can be succinctly formulated using order relations of this type. Thus, an absolute QI on $\Phi \in \text{Nat}(\mathcal{D}, \mathcal{A})$ may be expressed in terms of a subfunctor $\mathcal{F} \subset \mathcal{D}$ and a field $\mathcal{Q}^a \in \text{Nat}(\mathcal{F}, \mathcal{A})$ such that

$$\Phi|_{\mathcal{F}} \geq -\mathcal{Q}^a$$

with respect to the order relation on $\text{Nat}(\mathcal{F}, \mathcal{A})$, where $-\mathcal{Q}^a$ has components $(-\mathcal{Q}^a)_M(f) = -\mathcal{Q}_M^a(f)$. Note that in all QEIs currently known, $\mathcal{F}(M)$ is a convex subset of a vector space, but not a vector space in its own right, and $f \mapsto \mathcal{Q}_M^a(f)$ is nonlinear. Similarly, a DQI on Φ may be written in the form

$$\Delta\Phi|_{\mathcal{F}} \geq -\mathcal{Q}^d,$$

where $\mathcal{Q}^d \in \text{Nat}(\langle \mathcal{F}, \mathcal{S}^{\text{op}} \rangle, \mathcal{A})$ and $\Delta\Phi \in \text{Nat}(\langle \mathcal{D}, \mathcal{S}^{\text{op}} \rangle, \mathcal{A})$ is defined by

$$(\Delta\Phi)_M(f, \omega) = \Phi_M(f) - \omega(\Phi_M(f))\mathbf{1}_{\mathcal{A}(M)}$$

and is natural owing to the identity

$$\begin{aligned} (\Delta\Phi)_N(\psi_*f, \omega) &= \Phi_N(\psi_*f) - \omega(\Phi_N(\psi_*f))\mathbf{1}_{\mathcal{A}(N)} \\ &= \alpha_\psi(\Phi_M(f) - \alpha_\psi^*\omega(\Phi_M(f))\mathbf{1}_{\mathcal{A}(M)}) \\ &= \alpha_\psi((\Delta\Phi)_M(f, \alpha_\psi^*\omega)) \end{aligned}$$

holding whenever $\psi : M \rightarrow N$ in Man , for all $f \in \mathcal{D}(M)$ and $\omega \in \mathcal{S}(N)$.

In the rest of this section, we will set order relations of this type in a broader context, which will lead naturally to notions of a numerical range and spectrum of a locally covariant field, and also to an algebra of fields abstracted from particular spacetimes or smearings. This appears to provide a new way to analyse locally covariant fields, in which all constructions are automatically natural (i.e., covariant). The discussion of numerical range remains reasonably close to the subject of quantum inequalities; the spectrum is perhaps less immediately relevant, but its elementary theory is developed to demonstrate that it has the usual relationship to numerical range, and to illustrate the potential for covariant functional calculus of quantum fields.

Before we embark on this, it is necessary to dispose of a set-theoretical problem. The category Man is a large category: that is, obj Man is not a small set¹¹. The Whitney embedding theorem provides the standard solution to this problem, as it asserts that every smooth manifold of dimension d may be embedded as a smooth submanifold of \mathbb{R}^{2d+1} . Thus the collection of isomorphism equivalence classes in the category of smooth manifolds of dimension d may be identified with a subset of the power set of \mathbb{R}^{2d+1} , and is therefore a small set. The objects of Man are distinguished further by their metrics and (time-)orientations but, as it is easy to see that these are drawn from small sets, the argument extends straightforwardly to show that the isomorphism equivalence classes in Man form a small set, i.e., Man is an essentially small category. We now choose one representative from each isomorphism class, to obtain a small set \mathfrak{M} of ‘basic spacetimes’. The precise choice of these representatives will never be important.

Now note that any natural transformation $\Phi : \mathcal{D} \rightarrow \mathcal{A}$ is completely determined by its components Φ_M for $M \in \mathfrak{M}$. For each $M \in \text{Man}$ there is a unique $\widetilde{M} \in \mathfrak{M}$ for which $M \xrightarrow{\psi} \widetilde{M}$, with ψ

¹¹Given a manifold M , any set isomorphic to M (as a set) may be given the structure of a manifold by utilising the isomorphism, whereupon it is isomorphic to M as a manifold. Accordingly, obj Man must be at least as large as the set of all small sets of cardinality \aleph_c , and is therefore a large set.

a \mathbf{Man} -isomorphism. Thus $\alpha_\psi : \mathcal{A}(\mathbf{M}) \rightarrow \mathcal{A}(\widetilde{\mathbf{M}})$ is a \mathbf{TAlg} -isomorphism and $\Phi_{\mathbf{M}} = \alpha_\psi^{-1} \circ \Phi_{\widetilde{\mathbf{M}}} \circ \psi_*$ because Φ is natural. It follows that $\mathbf{Nat}(\mathcal{D}, \mathcal{A})$ is a small set, as it is isomorphic to a subset of the Cartesian product over $\mathbf{M} \in \mathfrak{M}$ of the set of functions from $\mathcal{D}(\mathbf{M})$ to $\mathcal{A}(\mathbf{M})$.

5.1 Numerical range

We begin by defining the numerical range relative to a given state space. If \mathcal{A} is a topological $*$ -algebra and $S \subset \mathcal{A}_{+,1}^*$ is convex, we define the numerical range of $A \in \mathcal{A}$, relative to S , by

$$N_{\mathcal{A},S}(A) = \text{cl}\{\omega(A) : \omega \in S\},$$

where cl denotes the closure in the topology of \mathbb{C} . Owing to convexity of S , $N_{\mathcal{A},S}(A)$ is convex for all A .

The numerical range is a well-known tool in the theory of quadratic forms in Hilbert spaces (and, in particular, matrix theory) [29]. A corresponding theory in Banach- $*$ -algebras is described in [2] (see also [42]). We have generalised this to \mathbf{TAlg} and permitted a restricted space of states; in addition, we differ from the references mentioned by taking the closure in our definition. This is convenient in our case, as shown by the following lemma.

Lemma 5.1 *Suppose $\mathcal{A}, \mathcal{B} \in \mathbf{TAlg}$. If $\alpha : \mathcal{A} \rightarrow \mathcal{B}$ is a faithful, unit-preserving $*$ -homomorphism and $T \subset \mathcal{B}_{+,1}^*$ obeys (i) $\alpha^*T \subset S$ and (ii) α^*T has the same weak- $*$ closure as S in $\mathcal{A}_{+,1}^*$, then we have*

$$N_{\mathcal{B},T}(\alpha(A)) = N_{\mathcal{A},S}(A)$$

for all $A \in \mathcal{A}$.

Proof: Fix $A \in \mathcal{A}$. Given any $\omega \in T$, it is clear that $\omega(\alpha(A)) = \alpha^*\omega(A) \in N_{\mathcal{A},S}(A)$ and hence that the left-hand side is contained in the right. On the other hand, choose any $\omega \in S$; then there exists a sequence $\omega_n \in T$ such that $\omega_n(\alpha(A)) = \alpha^*\omega_n(A) \rightarrow \omega(A)$, from which the reverse inclusion follows. \square

Now return to locally covariant QFT \mathcal{A} , with states \mathcal{S} satisfying local physical equivalence. Writing $2^{\mathbb{C}} : \mathbf{Man} \rightarrow \mathbf{Set}$ for the constant functor that assigns the power set of \mathbb{C} to each $\mathbf{M} \in \mathbf{Man}$ and the identity morphism $\text{id}_{2^{\mathbb{C}}}$ to each morphism of \mathbf{Man} , Lemma 5.1 entails that the maps $N_{\mathbf{M}}(A) = N_{\mathcal{A}(\mathbf{M}), \mathcal{S}(\mathbf{M})}(A)$ constitute a natural transformation $N : \mathcal{A} \rightarrow 2^{\mathbb{C}}$ expressed by commutativity of the diagram

$$\begin{array}{ccccc} \mathbf{M} & \mathcal{A}(\mathbf{M}) & \xrightarrow{N_{\mathbf{M}}} & 2^{\mathbb{C}} & \\ \psi \downarrow & \alpha(\psi) \downarrow & & \downarrow \text{id}_{2^{\mathbb{C}}} & \\ \mathbf{N} & \mathcal{A}(\mathbf{N}) & \xrightarrow{N_{\mathbf{N}}} & 2^{\mathbb{C}} & \end{array}.$$

Moreover, if we have a field $\Phi : \mathcal{D} \rightarrow \mathcal{A}$, then $N(\Phi)_{\mathbf{M}}(f) = N_{\mathbf{M}}(\Phi_{\mathbf{M}}(f))$ defines a natural transformation $N(\Phi) : \mathcal{D} \rightarrow 2^{\mathbb{C}}$.

We may use the numerical range to rephrase the construction of the sharp state-independent AQI of Prop. 4.4: we have $\tilde{\mathcal{Q}}_{\mathbf{M}}^a(f) = -\inf N(\Phi)_{\mathbf{M}}(f)$, which we could write as $\tilde{\mathcal{Q}}^a = \inf N(\Phi)$.

For future reference, we note the following. If $\psi : \mathbf{M} \rightarrow \mathbf{N}$ is an isomorphism in \mathbf{Man} , then

$$\bigcup_{f \in \mathcal{D}(\mathbf{M})} N(\Phi)_{\mathbf{M}}(f) = \bigcup_{f \in \mathcal{D}(\mathbf{M})} N(\Phi)_{\mathbf{N}}(\psi_* f) = \bigcup_{f \in \mathcal{D}(\mathbf{N})} N(\Phi)_{\mathbf{N}}(f), \quad (16)$$

where we have used the fact that $N(\Phi)$ is natural and the fact that $\psi_* : \mathcal{D}(\mathbf{M}) \rightarrow \mathcal{D}(\mathbf{N})$ is an isomorphism. Accordingly, these sets are spacetime invariants.

The order relation (15) may now be expressed as follows: $\Phi \geq 0$ if and only if $N(\Phi)_{\mathbf{M}}(f) \subset [0, \infty)$ for all $\mathbf{M} \in \mathbf{Man}$ and $f \in \mathcal{D}(\mathbf{M})$, and $\Phi \geq \Psi$ when $\Phi - \Psi \geq 0$. Here $\Phi - \Psi$ has components $(\Phi - \Psi)_{\mathbf{M}}(f) = \Phi_{\mathbf{M}}(f) - \Psi_{\mathbf{M}}(f)$ and is obviously a natural transformation between \mathcal{D} and \mathcal{A} .

Lemma 5.2 $\Phi \geq 0$ if and only if

$$\nu(\Phi) \stackrel{\text{def}}{=} \text{cl co} \left(\bigcup_{\mathbf{M} \in \mathfrak{M}} \bigcup_{f \in \mathcal{D}(\mathbf{M})} N(\Phi)_{\mathbf{M}}(f) \right)$$

is contained in $[0, \infty)$, where co is the operation of forming the convex hull.

Remark: In Mac Lane's description of category theory founded on a single universe [37], it is permissible to index a union over a small set, which is why we have used \mathfrak{M} instead of the large set obj Man . It follows from the proof that $\nu(\Phi)$ is independent of the particular choice of basic manifolds. In fact, the set-theoretical problem is not at all severe, because all the sets in the union are subsets of \mathbb{C} , so we could write

$$\nu(\Phi) = \text{cl co} \{ z \in \mathbb{C} : \exists \mathbf{M} \in \text{Man}, f \in \mathcal{D}(\mathbf{M}) \text{ s.t. } z = N(\Phi)_{\mathbf{M}}(f) \},$$

which is a legitimate subset selection within ZFC (see, e.g., Sect. I.5 of [36]). However, it is convenient to be able to use the union notation freely without abuse.

Proof: As $[0, \infty)$ is closed and convex it is enough to check that $\Phi \geq 0$ if and only if the union in parentheses is contained in $[0, \infty)$. But (16) and the fact that \mathfrak{M} contains a representative of every isomorphism class in Man , show that this is equivalent to the condition that $N(\Phi)_{\mathbf{M}}(f) \subset [0, \infty)$ for all $\mathbf{M} \in \text{Man}$, $f \in \mathcal{D}(\mathbf{M})$, which is the condition that $\Phi \geq 0$. \square

Note that $\nu(\Phi)$ cannot expand, and may contract, if \mathcal{D} is replaced by one of its subfunctors. Indeed, in many circumstances it may be necessary to make a replacement like this in order to cut the numerical range down from all of \mathbb{C} or \mathbb{R} .

In the previous result, the formation of the closed convex hull was redundant, but defining $\nu(\Phi)$ in the above way has the advantage that $\nu(\Phi)$ may be regarded as a numerical range in its own right. Notice that the set of natural transformations $\text{Nat}(\mathcal{D}, \mathcal{A})$ may be given the structure of a $*$ -algebra, with sums and products defined pointwise, i.e.,

$$\begin{aligned} \Phi_{\mathbf{M}}^*(f) &= (\Phi_{\mathbf{M}}(f))^* \\ (\lambda\Phi + \mu\Psi)_{\mathbf{M}}(f) &= \lambda\Phi_{\mathbf{M}}(f) + \mu\Psi_{\mathbf{M}}(f) \\ (\Phi\Psi)_{\mathbf{M}}(f) &= \Phi_{\mathbf{M}}(f)\Psi_{\mathbf{M}}(f), \end{aligned}$$

which are clearly both associative and distributive. There is a unit $\mathbf{1} : \mathcal{D} \rightarrow \mathcal{A}$ with components $\mathbf{1}_{\mathbf{M}}(f) = \mathbf{1}_{\mathcal{A}(\mathbf{M})}$, and we may endow the algebra with the topology of pointwise convergence, i.e., a net Φ_α converges to Ψ if we have $\Phi_\alpha_{\mathbf{M}}(f) \rightarrow \Psi_{\mathbf{M}}(f)$ for all $\mathbf{M} \in \text{Man}$, $f \in \mathcal{D}(\mathbf{M})$. We denote the resulting unital topological $*$ -algebra by $\mathcal{F}(\mathcal{D}, \mathcal{A})$.

Next, observe that each $(\mathbf{M}, f, \omega) \in \text{Man} \times \mathcal{D}(\mathbf{M}) \times \mathcal{S}(\mathbf{M})$ induces a linear functional $\xi_{\mathbf{M}, f, \omega}$ on $\mathcal{F}(\mathcal{D}, \mathcal{A})$ by

$$\xi_{\mathbf{M}, f, \omega}(\Phi) = \omega(\Phi_{\mathbf{M}}(f)),$$

which is clearly continuous in Φ , positive [$\xi_{\mathbf{M}, f, \omega}(\Phi^*\Phi) = \omega(\Phi_{\mathbf{M}}(f)^*\Phi_{\mathbf{M}}(f)) \geq 0$] and normalised [$\xi_{\mathbf{M}, f, \omega}(\mathbf{1}) = \omega(\mathbf{1}_{\mathcal{A}(\mathbf{M})}) = 1$]. If we define $S(\mathcal{D}, \mathcal{A})$ to consist of all finite convex combinations of states of this type, it is then immediate that

$$\nu(\Phi) = N_{\mathcal{F}(\mathcal{D}, \mathcal{A}), S(\mathcal{D}, \mathcal{A})}(\Phi).$$

Thus, a field Φ is positive, i.e., $\Phi \geq 0$, if and only if its numerical range in $\mathcal{F}(\mathcal{D}, \mathcal{A})$, relative to state space $S(\mathcal{D}, \mathcal{A})$ is contained in $[0, \infty)$. Note that the state space $S(\mathcal{D}, \mathcal{A})$ contains states which are mixtures of states associated with the theory on *different* spacetimes.

An important question is under what circumstances the infimum of the numerical range of a field Φ is attained. The following result shows that this cannot be the case [except for trivial situations] for any state ω which is separating for linear combinations of Φ and the identity field, in the sense that $\omega((\Phi_{\mathbf{M}}(f) + \mu\mathbf{1}_{\mathcal{A}(\mathbf{M})})^*(\Phi_{\mathbf{M}}(f) + \mu\mathbf{1}_{\mathcal{A}(\mathbf{M})})) = 0$ for some $\mu \in \mathbb{C}$ implies that $\Phi_{\mathbf{M}}(f) = -\mu\mathbf{1}_{\mathcal{A}(\mathbf{M})}$. In situations where a separating vacuum state exists (e.g., from a Reeh-Schlieder property) then the result shows that there must be states with expectation values for $\Phi_{\mathbf{M}}$ below that of the vacuum state. The argument is based on the proof of Lemma 1 of [11].

Proposition 5.3 *Suppose $\nu(\Phi) \subset [\nu_0, \infty)$, and that \mathcal{S} is closed under operations induced by \mathcal{A} . Suppose further that $\omega \in \mathcal{S}(\mathbf{M})$ is separating for linear combinations of Φ and the identity field. If $\Phi_{\mathbf{M}}(f) = \Phi_{\mathbf{M}}(f)^*$ obeys $\omega(\Phi_{\mathbf{M}}(f)) = \nu_0$, then $\Phi_{\mathbf{M}}(f) = \nu_0 \mathbf{1}_{\mathcal{A}(\mathbf{M})}$.*

Proof: The field $\Psi = \Phi - \nu_0 \mathbf{1}$ is positive, so we have a semidefinite sesquilinear form $(A, B) \mapsto \omega(A^* \Psi_{\mathbf{M}}(f) B)$ and hence a Cauchy–Schwarz inequality

$$|\omega(A^* \Psi_{\mathbf{M}}(f) B)|^2 \leq \omega(B^* \Psi_{\mathbf{M}}(f) B) \omega(A^* \Psi_{\mathbf{M}}(f) A).$$

Setting $B = \mathbf{1}_{\mathcal{A}(\mathbf{M})}$ and $A = \Psi_{\mathbf{M}}(f)$, we deduce that $\omega(\Psi_{\mathbf{M}}(f)^* \Psi_{\mathbf{M}}(f)) = 0$ (because $\omega(\Psi_{\mathbf{M}}(f)) = 0$) and hence that $\Phi_{\mathbf{M}}(f) = \nu_0 \mathbf{1}_{\mathcal{A}(\mathbf{M})}$ using the separating property. \square

The algebra $\mathcal{F}(\mathcal{D}, \mathcal{A})$ is of interest in its own right. It consists of the locally covariant fields of the theory, but abstracted from particular smearings in particular spacetimes (by virtue of knowing about all possible smearings in all possible spacetimes). Constructions conducted in this algebra and related structures are automatically natural – a point which we will develop in more detail for theories described by C^* -algebras. The following result is a consistency check on the ‘naturalness’ of the construction of $\mathcal{F}(\mathcal{D}, \mathcal{A})$.

Proposition 5.4 *Suppose $(\mathcal{D}_i, \mathcal{A}_i)$, for $i = 1, 2$, are equivalent in the sense that there are natural transformations $\alpha : \mathcal{A}_1 \rightarrow \mathcal{A}_2$ and $\delta : \mathcal{D}_1 \rightarrow \mathcal{D}_2$ with each $\alpha_{\mathbf{M}}$ and $\delta_{\mathbf{M}}$ an isomorphism in **TA**lg and **Set**, respectively. Then each $\Phi \in \text{Nat}(\mathcal{D}_1, \mathcal{A}_1)$ induces a natural transformation $\alpha \circ \Phi \circ \delta^{-1} : \mathcal{D}_2 \rightarrow \mathcal{A}_2$, and the map $\iota_{\delta, \alpha} \Phi \mapsto \alpha \circ \Phi \circ \delta^{-1}$ is an isomorphism of $\mathcal{F}(\mathcal{D}_1, \mathcal{A}_1)$ and $\mathcal{F}(\mathcal{D}_2, \mathcal{A}_2)$ in **TA**lg.*

Proof: Compositions of natural transformations are natural, so $\iota_{\delta, \alpha}(\Phi) \in \mathcal{F}(\mathcal{D}_2, \mathcal{A}_2)$ for each $\Phi \in \mathcal{F}(\mathcal{D}_1, \mathcal{A}_1)$. The fact that $\iota_{\delta, \alpha}$ respects the $*$ -algebraic operations and preserves the unit follows from the fact that each $\alpha_{\mathbf{M}}$ is a $*$ -homomorphism. Continuity holds because $\Phi_{\nu} \rightarrow \Psi$ implies that $(\iota_{\delta, \alpha} \Phi_{\nu})_{\mathbf{M}}(f) = \alpha_{\mathbf{M}}(\Phi_{\nu, \mathbf{M}}(\delta_{\mathbf{M}}^{-1}(f))) \rightarrow \alpha_{\mathbf{M}}(\Psi_{\mathbf{M}}(\delta_{\mathbf{M}}^{-1}(f))) = (\iota_{\delta, \alpha} \Psi)_{\mathbf{M}}(f)$ for all $\mathbf{M} \in \mathbf{Man}$ and $f \in \mathcal{D}_2(\mathbf{M})$, so $\iota_{\delta, \alpha} \Phi_{\nu} \rightarrow \iota_{\delta, \alpha} \Psi$. Since $\iota_{\delta, \alpha}$ has the obvious inverse $\iota_{\delta^{-1}, \alpha^{-1}}$ with the same properties, we conclude that it is a **TA**lg isomorphism. \square

As a digression, we mention that there are other possible algebraic combinations of fields. In particular, one may define a bi-local product \odot , mapping $(\Phi, \Psi) \in \text{Nat}(\mathcal{D}, \mathcal{A}) \times \text{Nat}(\mathcal{D}, \mathcal{A})$ to $\Phi \odot \Psi \in \text{Nat}(\langle \mathcal{D}, \mathcal{D} \rangle, \mathcal{A})$ such that

$$(\Phi \odot \Psi)_{\mathbf{M}}(f, g) = \Phi_{\mathbf{M}}(f) \Psi_{\mathbf{M}}(g)$$

and similarly bi-local sums and n -local sums and products. The resulting algebraic structures seem to offer compact, manifestly natural, expressions of commutation relations, and might be worthy of further study.

5.2 Spectrum

We now specialise to the case of theories where each $\mathcal{A}(\mathbf{M})$ is a C^* -algebra and each α_{ψ} is a faithful, unit-preserving C^* -morphism. In this context it is natural to restrict to a $*$ -subalgebra of $\mathcal{F}(\mathcal{D}, \mathcal{A})$, which we denote $\mathcal{F}^{\infty}(\mathcal{D}, \mathcal{A})$, consisting of those $\Phi \in \mathcal{F}(\mathcal{D}, \mathcal{A})$ for which

$$\|\Phi\| \stackrel{\text{def}}{=} \sup_{\mathbf{M} \in \mathfrak{M}} \sup_{f \in \mathcal{D}(\mathbf{M})} \|\Phi_{\mathbf{M}}(f)\|_{\mathcal{A}(\mathbf{M})}$$

is finite. Note that this is independent of the choice of basic spacetimes in \mathfrak{M} , because the inner supremum is constant on any isomorphism class in **Man**.

Proposition 5.5 *$\mathcal{F}^{\infty}(\mathcal{D}, \mathcal{A})$ is a C^* -algebra when equipped with the norm $\|\cdot\|$.*

Proof: It is enough to check that $\mathcal{F}^{\infty}(\mathcal{D}, \mathcal{A})$ is complete and that $\|\cdot\|$ has the C^* -property, as it is clear that $\mathcal{F}^{\infty}(\mathcal{D}, \mathcal{A})$ is a $*$ -algebra. To check completeness, note that any Cauchy sequence

Φ_n in $\mathcal{F}^\infty(\mathcal{D}, \mathcal{A})$ induces Cauchy sequences $\Phi_n \mathbf{M}(f)$ in $\mathcal{A}(\mathbf{M})$ for each $\mathbf{M} \in \mathbf{Man}$, $f \in \mathcal{D}(\mathbf{M})$. Denoting the corresponding limit by $\Phi_{\mathbf{M}}(f)$, one need only check that $\alpha_\psi(\Phi_{\mathbf{M}}(f)) = \Phi_{\mathbf{N}}(\psi_* f)$ (using continuity of α_ψ) to show that $\Phi_{\mathbf{M}} : f \mapsto \Phi_{\mathbf{M}}(f)$ form the components of a natural transformation $\Phi : \mathcal{D} \rightarrow \mathcal{A}$. As Φ_n is Cauchy, there exists m such that $\|\Phi_n \mathbf{M}(f) - \Phi_m \mathbf{M}(f)\|_{\mathcal{A}(\mathbf{M})} < 1$ for all $\mathbf{M} \in \mathbf{Man}$, $f \in \mathcal{D}(\mathbf{M})$ and $n \geq m$. From this it follows (taking $n \rightarrow \infty$) that $\|\Phi_{\mathbf{M}}(f)\| \leq 1 + \|\Phi_m \mathbf{M}(f)\| \leq 1 + \|\Phi_m\|$, so $\Phi \in \mathcal{F}^\infty(\mathcal{D}, \mathcal{A})$. Again, the Cauchy property for Φ_n implies that the convergence $\Phi_n \mathbf{M}(f) \rightarrow \Phi_{\mathbf{M}}(f)$ occurs uniformly in \mathbf{M} and f , so $\Phi_n \rightarrow \Phi$ in $\mathcal{F}^\infty(\mathcal{D}, \mathcal{A})$. The C^* -property follows straightforwardly from the C^* -property of each $\|\cdot\|_{\mathcal{A}(\mathbf{M})}$. \square

In many circumstances, it may be necessary to replace \mathcal{D} by one of its subfunctors in order to obtain a nontrivial algebra. Thus, for example, we might restrict to the unit ball with respect to a semi-norm on $\mathcal{D}(\mathbf{M})$, to keep the supremum over $\mathcal{D}(\mathbf{M})$ bounded for certain fields of interest. As an example of a nontrivial algebra $\mathcal{F}^\infty(\mathcal{D}, \mathcal{A})$, let $\mathcal{D}(\mathbf{M}) = C_0^\infty(\mathbf{M})$ and \mathcal{A} be the theory consisting of Weyl algebras of the free scalar field. Then $\mathcal{F}^\infty(\mathcal{D}, \mathcal{A})$ clearly contains the ‘Weyl field’ $W : \mathcal{D} \rightarrow \mathcal{A}$, defined so that $W_{\mathbf{M}}(f)$ is the Weyl generator associated with the test function $f \in \mathcal{D}(\mathbf{M})$ [informally, $W_{\mathbf{M}}(f) = e^{i\varphi_{\mathbf{M}}(f)}$].

The numerical range of fields in $\mathcal{F}^\infty(\mathcal{D}, \mathcal{A})$ may be defined as before, relative to the state space $S(\mathcal{D}, \mathcal{A})$ [which is also a state space for $\mathcal{F}^\infty(\mathcal{D}, \mathcal{A})$]. But we can now also invoke the spectrum, which is guaranteed to be well-behaved in the C^* -setting.

Let $\text{Sp}_{\mathcal{A}}(A)$ denote the spectrum of an element A of C^* -algebra \mathcal{A} . If $\alpha : \mathcal{A} \rightarrow \mathcal{B}$ is a unit-preserving faithful $*$ -homomorphism between C^* -algebras \mathcal{A} and \mathcal{B} , then $\text{Sp}_{\mathcal{B}}(\alpha(A)) = \text{Sp}_{\mathcal{A}}(A)$ for all $A \in \mathcal{A}$.¹² As with the numerical range, this entails the existence of a natural mapping $\text{Sp} : \mathcal{A} \rightarrow 2^{\mathbb{C}}$ expressed by commutativity of the diagram

$$\begin{array}{ccc} \mathbf{M} & \mathcal{A}(\mathbf{M}) & \xrightarrow{\text{Sp}_{\mathbf{M}}} 2^{\mathbb{C}} \\ \psi \downarrow & \alpha(\psi) \downarrow & \downarrow \text{id}_{2^{\mathbb{C}}} \\ \mathbf{N} & \mathcal{A}(\mathbf{N}) & \xrightarrow{\text{Sp}_{\mathbf{N}}} 2^{\mathbb{C}} \end{array} .$$

Hence, composing with any field $\Phi : \mathcal{D} \rightarrow \mathcal{A}$, we obtain a natural map $\text{Sp}(\Phi) : \mathcal{D} \rightarrow 2^{\mathbb{C}}$ such that

$$\text{Sp}(\Phi)_{\mathbf{M}}(f) = \text{Sp}_{\mathcal{A}(\mathbf{M})}(\Phi_{\mathbf{M}}(f)) .$$

In addition, we may also consider the spectrum of each $\Phi \in \mathcal{F}^\infty(\mathcal{D}, \mathcal{A})$, which we denote $\sigma(\Phi)$ for short. In contrast to $\text{Sp}(\Phi)$, this is not a natural transformation, but simply a subset of \mathbb{C} . Nonetheless, there is a relation between the two.

Proposition 5.6 *$\sigma(\Phi)$ and $\text{Sp}(\Phi)$ are related by*

$$\sigma(\Phi) = \text{cl} \left(\bigcup_{\mathbf{M} \in \mathfrak{M}} \bigcup_{f \in \mathcal{D}(\mathbf{M})} \text{Sp}(\Phi)_{\mathbf{M}}(f) \right) . \quad (17)$$

Proof: Suppose $\lambda \notin \sigma(\Phi)$. Then there exists $\Psi \in \mathcal{F}^\infty(\mathcal{D}, \mathcal{A})$ such that

$$\Psi(\lambda \mathbf{1} - \Phi) = (\lambda \mathbf{1} - \Phi)\Psi = \mathbf{1} ,$$

which entails that λ belongs to the resolvent set of every $\Phi_{\mathbf{M}}(f)$. Accordingly, we see that the right-hand side of (17) is contained in the left (as $\sigma(\Phi)$ is closed). Conversely, if λ does not belong to the right-hand side of (17), then there exists $\epsilon > 0$ for which the disc $\{\mu \in \mathbb{C} : |\mu - \lambda| < \epsilon\}$ lies in the resolvent set of every $\Phi_{\mathbf{M}}(f)$ (initially for $\mathbf{M} \in \mathfrak{M}$, but hence for all $\mathbf{M} \in \mathbf{Man}$). Setting $\Psi_{\mathbf{M}}(f) = (\lambda \mathbf{1}_{\mathcal{A}(\mathbf{M})} - \Phi_{\mathbf{M}}(f))^{-1}$, we have the uniform bound

$$\|\Psi_{\mathbf{M}}(f)\| \leq \epsilon^{-1} \quad \forall \mathbf{M} \in \mathbf{Man}, f \in \mathcal{D}(\mathbf{M}) . \quad (18)$$

¹²This follows using the fact that $\alpha(\mathcal{A})$ is C^* -subalgebra of \mathcal{B} (Prop. 2.3.1 of [3]) and because the spectrum of $\alpha(A)$, relative to $\alpha(\mathcal{A})$ is equal to its spectrum in \mathcal{B} - Prop. 2.2.7 of [3].

If $\psi : \mathbf{M} \rightarrow \mathbf{N}$ in \mathbf{Man} , we apply α_ψ to the equation $\Psi_{\mathbf{M}}(f)(\lambda \mathbf{1}_{\mathcal{A}(\mathbf{M})} - \Phi_{\mathbf{M}}(f)) = \mathbf{1}_{\mathcal{A}(\mathbf{M})} = (\lambda \mathbf{1}_{\mathcal{A}(\mathbf{M})} - \Phi_{\mathbf{M}}(f))\Psi_{\mathbf{M}}(f)$ to deduce that $\alpha_\psi \Psi_{\mathbf{M}}(f) = \Psi_{\mathbf{N}}(\psi_* f)$, and hence that the $\Psi_{\mathbf{M}}$ constitute a natural transformation $\Psi : \mathcal{D} \rightarrow \mathcal{A}$. The bound (18) then entails that $\Psi \in \mathcal{F}^\infty(\mathcal{D}, \mathcal{A})$, so $\lambda \mathbf{1} - \Phi$ is invertible in $\mathcal{F}(\mathcal{D}, \mathcal{A})$, completing the proof. \square

Fields in $\mathcal{F}^\infty(\mathcal{D}, \mathcal{A})$ may be manipulated according to functional calculus: for example, if $\Phi \in \mathcal{F}^\infty(\mathcal{D}, \mathcal{A})$ is normal and $\varphi : \sigma(\Phi) \rightarrow \mathbb{C}$ is continuous then there is an element $\varphi(\Phi) \in \mathcal{F}^\infty(\mathcal{D}, \mathcal{A})$, with $\sigma(\varphi(\Phi)) = \varphi(\sigma(\Phi))$. The field $\varphi(\Phi)$ is automatically covariant, and obeys $\varphi(\Phi)_{\mathbf{M}}(f) = \varphi(\Phi_{\mathbf{M}}(f))$. While the latter could also serve as a definition, we would need to check naturality. The advantage of using our algebra $\mathcal{F}^\infty(\mathcal{D}, \mathcal{A})$ is that naturality is automatic, so we have a manifestly covariant functional calculus.

We also obtain a new definition of a positive field, as one whose spectrum is positive, i.e., $\sigma(\Phi) \subset [0, \infty)$. Standard C^* -algebra theory entails that $\Phi \in \mathcal{F}^\infty(\mathcal{D}, \mathcal{A})$ is positive if and only if it is the square of another field; we also have that $\Phi^* \Phi$ is positive for any Φ .

We may easily recover one of the key properties of the numerical range, provided that S is sufficiently large.

Lemma 5.7 *Suppose \mathcal{A} is a C^* -algebra, and S is weak-* dense in $\mathcal{A}_{+,1}^*$. If $N_{\mathcal{A},S}(A)$ is contained in the real axis then the convex hull of the $\text{Sp}_{\mathcal{A}}(A)$ is*

$$\text{co Sp}_{\mathcal{A}}(A) = N_{\mathcal{A},S}(A).$$

In particular, this holds if S contains at least one state inducing a faithful representation of \mathcal{A} and is closed under operations induced by \mathcal{A} .

Proof: Using weak-* density and the fact that we have required the numerical range to be closed, we have $N_{\mathcal{A},S}(A) = N_{\mathcal{A},\mathcal{A}_{+,1}^*}(A)$. This is equal to the standard definition of the numerical range of A as in [2, 42] because the numerical range turns out to be closed for elements of C^* -algebras (Proposition 2.6.2(a) in [42]). The result then follows using the standard result for numerical range, e.g., Theorem 2.6.7(d) in [42]. The last statement follows by the proof of Lemma 4.2. \square

Proposition 5.8 *Suppose each $\mathcal{A}(\mathbf{M})$ is a C^* -algebra and each $\mathcal{S}(\mathbf{M})$ is closed under operations induced by $\mathcal{A}(\mathbf{M})$ and contains at least one state inducing a faithful GNS representation of $\mathcal{A}(\mathbf{M})$. If $\Phi \in \mathcal{F}^\infty(\mathcal{D}, \mathcal{A})$ has numerical range $\nu(\Phi) \subset \mathbb{R}$ then*

$$\text{co } \sigma(\Phi) = \nu(\Phi).$$

Proof: By Lemma 4.2 and Proposition 4.3, we know that each $\mathcal{S}(\mathbf{M})$ is weak-* dense in $\mathcal{A}(\mathbf{M})_{+,1}^*$ and that \mathcal{S} respects local physical equivalence. Hence we may define the numerical range $N(\Phi) : \mathcal{D} \rightarrow 2^{\mathbb{C}}$. Now $\nu(\Phi) \subset N_{\mathcal{F}^\infty(\mathcal{D}, \mathcal{F}), \mathcal{F}^\infty(\mathcal{D}, \mathcal{F})_{+,1}^*}(\Phi)$ because it is a numerical range over a subset of states. But the latter set is equal to $\text{co } \sigma(\Phi)$ by Lemma 5.7, so $\nu(\Phi) \subset \text{co } \sigma(\Phi)$, and is, in particular, bounded.

On the other hand, by Lemma 5.7 we have

$$\text{Sp}(\Phi)_{\mathbf{M}}(f) \subset \text{co Sp}_{\mathcal{A}(\mathbf{M})}(\Phi_{\mathbf{M}}(f)) = N_{\mathcal{A}(\mathbf{M}), \mathcal{S}(\mathbf{M})}(\Phi_{\mathbf{M}}(f)) = N(\Phi)_{\mathbf{M}}(f).$$

Taking the union over all $\mathbf{M} \in \mathfrak{M}$ and f , closing, and forming the convex hull, we obtain [also using Proposition 5.6]

$$\text{co } \sigma(\Phi) \subset \text{co cl} \left(\bigcup_{\mathbf{M} \in \mathfrak{M}} \bigcup_{f \in \mathcal{D}(\mathbf{M})} N(\Phi)_{\mathbf{M}}(f) \right) = \text{cl co} \left(\bigcup_{\mathbf{M} \in \mathfrak{M}} \bigcup_{f \in \mathcal{D}(\mathbf{M})} N(\Phi)_{\mathbf{M}}(f) \right) = \nu(\Phi),$$

because cl and co commute on bounded sets in \mathbb{R}^k (see e.g., Theorem 17.2 in [43], although this is essentially obvious in our 1-dimensional case). Hence $\text{co } \sigma(\Phi) = \nu(\Phi)$ as required. \square

An interesting point about this result is that $S(\mathcal{D}, \mathcal{A})$ contains sufficiently many states to guarantee the usual connection between spectrum and numerical range, even though we do not know whether it is weak-* dense in $\mathcal{F}^\infty(\mathcal{D}, \mathcal{A})_{+,1}^*$.

6 Conclusion

The main purpose of this paper has been to locate quantum (energy) inequalities within the categorical framework of local covariance developed by BFV. This has led us to a broader definition of QIs than has previously been adopted, because we allow for the possibility of state-dependent lower bounds. This seems natural from the categorical point of view and also appears to be needed in some specific instances, including the non-minimally coupled scalar field [22]. We have also given a first attempt to delineate when such a bound should be regarded as trivial, taking our inspiration from the sharp Gårding inequalities, and we have checked that our definitions respect covariance and are compatible with each other in various ways. In the process it has become clear that the property of local physical equivalence, isolated here for the first time, plays an important role in the analysis of locally covariant quantum field theories. We have also considered the broader question of the definition and basic properties of covariant numerical range and spectrum of local quantum fields, leading naturally to the abstract algebras of fields $\mathcal{F}(\mathcal{D}, \mathcal{A})$ and $\mathcal{F}^\infty(\mathcal{D}, \mathcal{A})$. As an application of some of our ideas, we have shown how information about spatially toroidal spacetimes can be used to infer properties of quantum field theory on Minkowski space.

Our work raises several questions for future study. Can one give a formal, locally covariant, definition of what it means for one field to be of ‘lower order’ than another? In Minkowski space one could appeal to H -bounds to provide a scale of fields, but in general one has no global Hamiltonian. It is hoped that the definition sketched in Sect. 3.1 can be developed to meet this end. More broadly, is the notion of triviality studied here a sufficiently stringent definition? If not, can a more refined version be found? This might well take the form of a grading on the elements of algebras such as $\mathcal{F}(\mathcal{D}, \mathcal{A})$. It is also necessary to investigate the local physical equivalence property in the context of known models. One would also like to make a more precise connection between QIs and the phase space properties of a theory, perhaps establishing them as precise analogues of the sharp Gårding inequalities. In turn, this raises the question of how the phase space of the theory may be controlled in the locally covariant setting. Above all, a key question is to determine what structural features of a locally covariant quantum field theory are sufficient to guarantee the existence of QEIs. It is hoped to return to these questions elsewhere.

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A The Wick square of the free scalar field

As a concrete example, we briefly consider a difference QI on the Wick square of the free scalar field of mass m obtained by the methods of [15]. In BFV, the free scalar field theory was explicitly expressed in terms of a functor $\mathcal{A} : \mathbf{Man} \rightarrow \mathbf{TAlg}$, and the free scalar field itself as a natural transformation $\varphi : \mathcal{D} \rightarrow \mathcal{A}$, with each $\mathcal{D}(\mathbf{M})$ equal to the smooth compactly supported complex-valued functions on \mathbf{M} . Hollands and Wald [32] constructed enlarged algebras $\mathcal{W}(\mathbf{M})$ that represent algebras of Wick polynomials in $\varphi_{\mathbf{M}}$, and these algebras may be regarded as the images on objects of a functor $\mathcal{W} : \mathbf{Man} \rightarrow \mathbf{TAlg}$. Each $\mathcal{A}(\mathbf{M})$ may be regarded as a sub- $*$ -algebra of $\mathcal{W}(\mathbf{M})$. The state space \mathcal{S} associated with \mathcal{W} is defined so that $\mathcal{S}(\mathbf{M})$ consists of all states on $\mathcal{W}(\mathbf{M})$ whose two-point functions obey the Hadamard condition (expressed as a certain condition on the wave-front set [34]), and whose truncated n -point functions for $n \neq 2$ are all smooth. The reader is referred to BFV and [32, 31] for more details; below, we restrict attention to the portion of the structure relevant to our discussion.

Let \mathcal{E}'_1 be the functor $\mathcal{E}'_1 : \mathbf{Man} \rightarrow \mathbf{Set}$ defined to act on any object $\mathbf{M} \in \mathbf{Man}$ so that

$$\mathcal{E}'_1(\mathbf{M}) = \{f \in \mathcal{E}'(\mathbf{M}) : \text{WF}(f) \cap \mathcal{V}_{\mathbf{M}} = \emptyset\},$$

where $\mathcal{E}'(\mathbf{M})$ is the usual space of compactly supported (scalar) distributions on \mathbf{M} , $\text{WF}(f)$ denotes the wave-front set of a distribution f and $\mathcal{V}_{\mathbf{M}} \subset T^*\mathbf{M}$ is the bundle of causal covectors on \mathbf{M} . Given any $\psi : \mathbf{M} \rightarrow \mathbf{N}$, we define ψ_* to be the induced push-forward of compactly supported distributions $\psi_* : \mathcal{E}'(\mathbf{M}) \rightarrow \mathcal{E}'(\mathbf{N})$; setting $\mathcal{E}'_1(\psi) = \psi_*$ (or more precisely its restriction to $\mathcal{E}'_1(\mathbf{M})$) it is easy to verify that \mathcal{E}'_1 is a functor as claimed, of which $\mathcal{D}(\mathbf{M})$ is a subfunctor.

The free scalar field φ extends to a natural transformation between \mathcal{E}'_1 and $\mathcal{W}(\mathbf{M})$; by contrast, the Wick square is not uniquely defined. Rather, there is a family of natural transformations between \mathcal{E}'_1 and \mathcal{W} with the property that, if φ^2 and $\tilde{\varphi}^2$ are any two members of the family, then there are real constants c_1 and c_2 such that

$$\varphi_{\mathbf{M}}^2(f) - \tilde{\varphi}_{\mathbf{M}}^2(f) = f(c_1 R_{\mathbf{M}} + c_2 m^2) \mathbf{1}_{\mathcal{W}(\mathbf{M})}$$

for all $f \in \mathcal{E}'_1(\mathbf{M})$, and each $\mathbf{M} \in \mathbf{Man}$. Here $R_{\mathbf{M}}$ is the Ricci scalar on \mathbf{M} . Any one of these natural transformations provides a valid definition of the Wick square: we henceforth suppose that one has been chosen, which we denote φ^2 . (For a proposal to fix the renormalisation constants on thermodynamic grounds, see [8]). Given any $\omega, \omega' \in \mathcal{S}(\mathbf{M})$, we also have

$$\omega(\varphi_{\mathbf{M}}^2(f)) - \omega'(\varphi_{\mathbf{M}}^2(f)) = \delta_2^*(\Lambda_{\omega} - \Lambda_{\omega'})(f), \quad (19)$$

where $\delta_2 : \mathbf{M} \rightarrow \mathbf{M} \times \mathbf{M}$ is defined by $\delta_2(p) = (p, p)$ and Λ_{ω} denotes the two-point function of ω , i.e., $\Lambda_{\omega}(p, p') = \omega(\varphi_{\mathbf{M}}(p)\varphi_{\mathbf{M}}(p'))$ in unsmeared notation; δ_2^* denotes the distributional pull-back by δ_2 .

Classically, of course, a squared field is pointwise nonnegative. In the quantum field theory, however, the quantised Wick square is capable of assuming negative values, which turn out to be constrained by a DQI. For each $\mathbf{M} \in \mathbf{Man}$, let $\mathcal{F}(\mathbf{M})$ be the set of $f \in \mathcal{E}'_1(\mathbf{M})$ such that

$$f(u) = \int_{\gamma} g(\tau)^2 u(\gamma(\tau)) d\tau \quad (u \in C^{\infty}(\mathbf{M})),$$

where I is an open interval of \mathbb{R} , $\gamma : I \rightarrow \mathbf{M}$ is a proper time parameterisation of a smooth, future-pointing timelike curve, and g is a smooth real-valued function, compactly supported in I and with no zeros of infinite order in the interior of its support. Given f , we may reconstruct I , γ and g up to reparameterisations which may be ignored in the following discussion (see [24] for details). The DQI obtained by the methods of [15] is

$$\omega(\varphi_{\mathbf{M}}^2(f)) - \omega_0(\varphi_{\mathbf{M}}^2(f)) \geq - \int_0^{\infty} \frac{d\alpha}{\pi} [(g \otimes g) \gamma_2^* \Lambda_{\omega_0}]^{\wedge}(-\alpha, \alpha) \stackrel{\text{def}}{=} - \tilde{\mathcal{Q}}_{\mathbf{M}}(f, \omega_0), \quad (20)$$

where γ_2^* denotes the distributional pull-back from $\mathbf{M} \times \mathbf{M}$ to \mathbb{R}^2 by the map $\gamma_2(\tau, \tau') = (\gamma(\tau), \gamma(\tau'))$ and the hat denotes a Fourier transform (see [24] for the conventions). In [24] arguments are given which show that the central member of (20) is independent of the particular parameterisation of f in terms of I , γ , and g ; these arguments also show that \mathcal{F} is a subfunctor of \mathcal{E}'_1 and that the covariance relation $\tilde{\mathcal{Q}}_{\mathbf{N}}(\psi_* f, \omega_0) = \tilde{\mathcal{Q}}_{\mathbf{M}}(f, \psi^* \omega_0)$ holds. Thus $\mathcal{Q}_{\mathbf{M}}(f, \omega_0) = \tilde{\mathcal{Q}}_{\mathbf{M}}(f, \omega_0) \mathbf{1}_{\mathcal{F}(\mathbf{M})}$ defines a natural map $\mathcal{Q} : \langle \mathcal{F}, \mathcal{S}^{\text{op}} \rangle \rightarrow \mathcal{W}$, establishing this bound as a locally covariant difference quantum inequality with respect to \mathcal{S} .

As well as providing a concrete example of the various functors and natural transformations involved in our framework, we also want to point out two features of this bound not discussed in [24]. First, we prove that it satisfies the condition (10). This is essentially the reverse of the argument in [15]: noting that

$$\tilde{\mathcal{Q}}_{\mathbf{M}}(f, \omega) - \tilde{\mathcal{Q}}_{\mathbf{M}}(f, \omega_0) = \int_0^{\infty} \frac{d\alpha}{\pi} [(g \otimes g) \gamma_2^* (\Lambda_{\omega} - \Lambda_{\omega_0})]^{\wedge}(-\alpha, \alpha),$$

we use the fact that $\Lambda_\omega - \Lambda_{\omega_0}$ is symmetric and smooth to rewrite the right-hand side as

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{d\alpha}{2\pi} [(g \otimes g) \gamma_2^* (\Lambda_\omega - \Lambda_{\omega_0})]^\wedge (-\alpha, \alpha) &= \int d\tau g(\tau)^2 (\Lambda_\omega - \Lambda_{\omega_0}) (\gamma_2(\tau)) \\ &= \omega(\varphi_M^2(f)) - \omega_0(\varphi_M^2(f)) \end{aligned}$$

as required, using the Fourier representation of the δ -function and (19). Accordingly, as shown in Sect. 3.2, $\mathcal{Q}_M(f) = \mathcal{Q}_M(f, \omega_0) - \omega_0(\varphi_M^2(f)) \mathbf{1}_{\mathcal{A}(M)}$ defines a locally covariant AQI, because the right-hand side is independent of $\omega_0 \in \mathcal{S}(M)$.

The second fact about our original DQI follows immediately from the first: namely the map $\omega_0 \mapsto \mathcal{Q}_M(f, \omega_0)$ is weak-* continuous for each $f \in \mathcal{F}(M)$; furthermore, $(\omega_0, \omega) \mapsto \omega(\mathcal{Q}_M(f, \omega_0))$ is weak-* continuous in ω_0 , uniformly in $\omega \in \mathcal{S}(M)$. This shows that this DQI obeys the continuity hypotheses required in Propositions 4.5 and 4.6(b).

B The non-minimally coupled scalar field

Recall that the non-minimally coupled field obeys the field equation $(\square + m^2 + \xi R)\varphi = 0$, where ξ is called the coupling constant; its stress-energy tensor differs from that of the minimally coupled ($\xi = 0$) theory by terms which permit the violation of the pointwise energy conditions even at the classical level. One may show that this renders it impossible to obtain state-independent QEIs [22]. Our purpose in this Appendix is to illustrate the discussion of state-dependent QEIs with an example to show the type of state-dependence that enters in the difference QEI obtained in [22] by the present author and Osterbrink for the case $0 \leq \xi \leq 1/4$.

Let $\mathcal{E}'_1 T_0^2 : \mathbf{Man} \rightarrow \mathbf{Set}$ be the functor assigning to each $M \in \mathbf{Man}$ the set of compactly supported rank-2 contravariant tensor distributions

$$\mathcal{E}'_1 T_0^2(M) = \{f \in \mathcal{E}'(T_0^2 M) : \text{WF}(f) \cap \mathcal{V}_M = \emptyset\},$$

with \mathcal{V}_M as in Appendix A, and $\mathcal{E}'_1 T_0^2$ acting on morphisms to give the corresponding push-forward. Then the stress-energy tensor of the theory may be regarded as a natural transformation $T : \mathcal{E}'_1 T_0^2 \rightarrow \mathcal{W}$, using the usual point-splitting prescriptions to define T_M in any particular spacetime M . (As with the Wick square, there is a family of possible different candidates, and we assume that one has been chosen.) We define a subfunctor $\mathcal{F} \subset \mathcal{E}'_1 T_0^2$ so that $\mathcal{F}(M)$ consists of those $f \in \mathcal{E}'_1 T_0^2(M)$ of the form

$$f(t) = \int_{\gamma} g(\tau)^2 \dot{\gamma}^a \dot{\gamma}^b t_{ab}|_{\gamma(\tau)} d\tau \quad (t \in C^\infty(T_2^0 M)), \quad (21)$$

where $\gamma : I \rightarrow M$ is a proper time parameterisation of a smooth, future-directed timelike *geodesic*, with $I \subset \mathbb{R}$ open, and $g \in C_0^\infty(I; \mathbb{R})$ obeying the same hypotheses as in Appendix A. The QEI obtained in [22] can be expressed in the form

$$\omega(T_M(f)) - \omega_0(T_M(f)) \geq -\omega(\mathcal{Q}_M(f, \omega_0)),$$

for states $\omega, \omega_0 \in \mathcal{S}(M)$ and any $f \in \mathcal{F}(M)$, where $\mathcal{Q} : \langle \mathcal{F}, \mathcal{S}^{\text{op}} \rangle \rightarrow \mathcal{W}$ is natural and has components $\mathcal{Q}_M(f) \in \mathcal{W}(M)$ of the form

$$\mathcal{Q}_M(f, \omega_0) = \tilde{\mathcal{Q}}_M(f, \omega_0) \mathbf{1}_{\mathcal{W}(M)} + \varphi_M^2(S_M(f)),$$

where φ^2 is the renormalised Wick square. Here, S_M is a component of a natural transformation $S : \mathcal{F} \rightarrow \mathcal{E}'_1$, defined so that

$$(S_M(f))(u) = f(uW_M) + 2\xi \int_{\gamma} \dot{g}(\tau)^2 u(\gamma(\tau)) d\tau \quad (u \in C^\infty(M)), \quad (22)$$

for f of the form (21) and the tensor $W_{\mathbf{M}}$ multiplying u is given by

$$W_{\mathbf{M}} = \xi \text{Ric}_{\mathbf{M}} - \frac{1}{2} \xi (1 - 4\xi) R_{\mathbf{M}} g_{\mathbf{M}}$$

where $\text{Ric}_{\mathbf{M}}$, $R_{\mathbf{M}}$ and $g_{\mathbf{M}}$ are the Ricci tensor, Ricci scalar and metric of \mathbf{M} . Note that $W_{\mathbf{M}}$ vanishes if $\xi = 0$ or \mathbf{M} is Ricci flat, and $S_{\mathbf{M}}(f) = 0$ for any f if $\xi = 0$, thus rendering the bound state-independent in this case. [In the above, φ^2 , \mathcal{W} , \mathcal{S} and \mathcal{E}_1^f are all as in Appendix A.] In addition, $\tilde{\mathcal{Q}}_{\mathbf{M}}(f)$ also depends on ξ , and reduces to a previously known state-independent bound if $\xi = 0$. Details appear in [22]. For our current purpose the main point is that the state-dependence is a smearing of the Wick square φ^2 , whereas the stress-energy tensor itself involves squares of derivatives of φ . This fits well with the analogy between nontrivial QEIs and the gain in derivatives exhibited by the Gårding inequalities. In [22] we also show that the bound is nontrivial in the sense described in this paper: for example, in thermal states on Minkowski space the lower bound scales like the square of the temperature while the smeared stress-energy tensor itself scales with the fourth power.

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